# Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will cover up to §5.1 in the text and will be comprehensive. The final exam is at 8:30 am, Monday, June 4, in the regular classroom.

1. Let $u(x, y),\ v(x, y)$ be continuously differentiable as functions of $(x, y)$ in a domain $\Omega$. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an $r_0$ (depending on $z_0$) such that

$$\int_{|z-z_0|=r} f(z)dz = 0,$$

   for all $r$ with $r < r_0$. Prove that $f$ is analytic in $\Omega$. Hint: Show that $f$ satisfies the Cauchy-Riemann equations in $\Omega$.

2. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \leq x \leq 1\}$. Find a bounded harmonic function $u$, defined in $D_2 - I$ such that $u$ does not extend to a harmonic function defined in all of $D_2$.

3. Find a conformal map from the region between the two lines $y = x$ and $y = x + 2$ to the upper half plane, which sends 0 to 0.

4. Find a function, $h(x, y)$, harmonic in $\{x > 0,\ y > 0\}$, such that

$$h(x, y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

5. The function
$$f(z) = \frac{1}{(z+1)(z+3)}$$
has three different Laurent series expansions in powers of $z$. Find them all and specify the annulus for each series.

6. Let
$$f(z) = \exp(z + \frac{1}{z})$$
Prove that
$$\text{Res}[f, 0] = \sum_{k=0}^{k=\infty} \frac{1}{k!(k+1)!}$$
Carefully justify all steps in your computation.

7. Suppose that $u$ is a harmonic function and $v$ is its conjugate harmonic function. Prove that $uv$ is harmonic.

8. Let $f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \dots$. Prove that the series converges absolutely and uniformly on compact sets to an analytic function if $\text{Re}(z) > 1$. Prove that $f(z) = (1 - 2^{1-z})\zeta(z)$ if $\text{Re}(z) > 1$. Prove that the series converges conditionally when $z = x$ is real and $0 < x < 1$. Prove that this statement is still true on the infinite strip $0 < \text{Re}(z) < 1$.

9. Let $f_n(z)$ be a sequence of analytic functions defined on an open connected set $\Omega$. Suppose the sequence converges uniformly on all compact subsets of $\Omega$. Use Morera's theorem to prove that the limit function is analytic. Show that if $f_n$ has no zeros for all $n$ and if $f$ is not identically zero then $f$ has no zeros.

10. Let $u(z)$ be harmonic in $\{z : 0 < |z| < 1\}$. Let $P = \int_{|z|=r} \frac{\partial u}{\partial n} ds$, where $0 < r < 1$. Show that $P$ does not depend on $r$. Prove that
$$u(z) = \frac{P}{2\pi} \log|z| + \text{Re}(f(z))$$
where $f$ is analytic in $\{z : 0 < |z| < 1\}$.

11. Prove that if $|z| < 1$

$$\lim_{n\to\infty} \prod_{k=0}^{k=n} (1 + z^{2^k}) = \frac{1}{1-z}$$

12. Find the Green's function for the region $\{z : |z-2| < 3\}$ with pole at 4.

13. The characteristic function, $\chi_S$ of a set is defined by

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases}$$

   (a) Let $I = [a,b] \subset \mathbf{R}$ be a compact interval. Prove that $\lim_{x\to\pm\infty} \widehat{\chi}_I(x) = 0$ by directly computing $\widehat{\chi}_I(x)$ and then computing the limit.

   (b) Let $\{I_j\}_{j=1}^n$ be a finite collection of compact intervals in $\mathbf{R}$ and let $\{c_j\}_{j=1}^n$ be a finite collection of constants. Let $g = \sum_{j=1}^{n} c_j \chi_{I_j}$. Using part (a), prove that $\lim_{x\to\pm\infty} \widehat{g}(x) = 0$. (This can be used to give an alternate proof of the Riemann-Lebesgue lemma.

14. There may be homework problems or example problems from the text on the final. Don't forget previous sample problems. The final will be comprehensive.