Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover up to section 2.6 in the text.

1. Suppose that v is the harmonic conjugate of u and u is the harmonic conjugate of v. Show that u and v must be constant.

2. Let a and c be real constants and let b be a complex constant. Suppose $a \neq 0$). Prove that $az\bar{z} + \bar{b}z + b\bar{z} + c = 0$ is the equation of a circle. Find its center and radius.

3. Let $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$) be an analytic function. Let $\alpha = (\cos\phi, \sin\phi)$ and $\beta = (-\sin\phi, \cos\phi)$.

   (a) Write a formula for the directional derivative $\frac{\partial u}{\partial \alpha}$ in terms of the components of α and the partial derivatives of u. Do the same for $\frac{\partial v}{\partial \beta}$.

   (b) Using the Cauchy-Riemann equations, show that
   $$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta}.$$
   By a similar calculation show that
   $$\frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha}.$$

   (c) Suppose that u and v are functions that satisfy the equations of part (b) for a fixed ϕ. Using those equations alone, show that u and v satisfy the Cauchy-Riemann equations (for partials with respect to x and y).

4. Let $z_j = e^{\frac{2\pi ij}{n}}$ denote the n roots of unity. Let $c_j = |1 - z_j|$ be the $n - 1$ chord lengths from 1 to the points $z_j, j = 1, \ldots .n - 1$. Prove that the product $c_1 \cdot c_2 \cdots c_{n-1} = n$. *Hint*: Consider $z^n - 1$.

5. Let $f(z) = \cos x + i \sin y$. Find the points at which f is complex differentiable. Find the points at which f is complex analytic.

6. Find the Laurent series of the function $\frac{1}{z-2}$ in the annulus $D = \{z : 1 < |z - 1| < \infty\}$.

7. Using the calculus of residues, compute

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 3} dx$$

8. (a) Prove that

$$\frac{1}{(1-z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} z^n$$

   (b) What is the radius of convergence of the series in part (a)?

9. Let $f(z) = \dfrac{p'(z)}{zp(z)}$, where $p(z) = \prod_{j=1}^{n}(z - z_j)$ and the z_j are distinct and different from 0. Find all the poles of f and compute the residues of f at these poles.

10. Let f be analytic within and on a simple closed curve Γ. Prove that $Re(\int_\Gamma \bar{f}(z) f'(z) dz) = 0$.

11. Compute $\displaystyle\int_{|z|=r} \frac{|dz|}{|z-a|^2}$, where $|a| \neq r$. Use the fact that on $\{|z| = r\}$, $|dz| = -ir\dfrac{dz}{z}$, and then use the Cauchy integral formula.

12. You will need to know the definitions of the following terms and statements of the following theorems.

    1. Absolute Value (Modulus) and Argument of a complex number
    2. $\lim_{z\to a} f(z)$
    3. Continuity
    4. Complex Derivative
    5. Cauchy-Riemann equations
    6. Harmonic Conjugate
    7. Complex Analytic
    8. Differentiability of Power Series
    9. Complex Exponential Function
    10. Complex Logarithm
    11. Cauchy's Integral Theorem
    12. Cauchy's Integral Formula
    13. Morera's Theorem
    14. Liouville's Theorem
    15. Isolated Singularities (types)
    16. Residues
    17. Residue Theorem
    18. Laurent Series

13. There may be homework problems or example problems from the text on the midterm.