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$V^i, R^i, I$  and  $D_1 \neq D_2^i$  for all  $i$ . Hence  $D_1 \notin R(G)$  and  $D_2 \notin R(G)$ . Finally  $HR^2 = V = R^2H$ . But  $H \neq V^i, R^i, D_1^i, D_2^i, I$  and  $R^2 \neq H^i$  for all  $i$ . Hence  $H \notin R(G)$ . Thus  $AC(G): I, R, R^2$ , and  $R^3$ .

The converses of Theorem 2 and Theorem 4 are false as Example 2 indicates. Finally the  $AC(G)$  is, in general nontrivial, as is demonstrated by Example 3.

In a later paper, the author hopes to obtain results concerning  $G/AC(G)$ , some relations between the center and the anticenter, and to explore more fully the effect of isomorphism and homomorphism on the anticenter.

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## CLASSROOM NOTES

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### A DERIVATION OF $n$ -DIMENSIONAL SPHERICAL COORDINATES

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An instructive example in linear algebra is the derivation of  $n$ -dimensional spherical coordinates without appealing to geometric intuition. The method of derivation is based on concepts from linear algebra; namely, bases of a vector space, scalar product, angle between vectors and projection of a vector onto a subspace. Spherical coordinates in  $n$ -dimensions are a generalization of the usual three-dimensional spherical coordinates and are particularly useful in evaluating certain integrals taken over the surface of an  $n$ -dimensional sphere. Later we shall give an example of such an integration.

Let  $E_n$  denote real  $n$ -dimensional euclidean space. Vectors in  $E_n$  will be denoted by bold-faced letters. If  $\mathbf{x}$  and  $\mathbf{y}$  are two vectors in  $E_n$  with components  $\xi_j$  and  $\eta_j$ ,  $j = 1, \dots, n$ , respectively, we define the scalar product of  $\mathbf{x}$  and  $\mathbf{y}$  by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^n \xi_j \eta_j.$$

The nonnegative number  $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$  is called the norm of  $\mathbf{x}$ . The angle between  $\mathbf{x}$  and  $\mathbf{y}$  is defined by  $\cos \phi = \mathbf{x} \cdot \mathbf{y} / \|\mathbf{x}\| \|\mathbf{y}\|$ , where  $\phi$  is restricted to the range  $0 \leq \phi \leq \pi$ . A set of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  is an orthonormal set in  $E_n$  if  $\mathbf{x}_i \cdot \mathbf{x}_j = 0$  or 1 accordingly as  $i \neq j$  or  $i = j$ . Any set of  $n$  orthonormal vectors forms a basis for  $E_n$ .

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be any orthonormal basis in  $E_n$ . Let  $\mathbf{x}$  be any vector on the  $n$ -dimensional sphere of radius  $r$  about the origin, that is,  $\|\mathbf{x}\| = r$ . If  $\mathbf{x} = \sum_{i=1}^n \xi_i \mathbf{e}_i$ ; then  $\|\mathbf{x}\|^2 = \sum_{i=1}^n \xi_i^2$ . If  $\theta_i$  is the angle between  $\mathbf{x}$  and  $\mathbf{e}_i$  then  $\xi_i = \mathbf{x} \cdot \mathbf{e}_i = r \cos \theta_i$ . Hence  $\mathbf{x} = \sum_{i=1}^n r \cos \theta_i \mathbf{e}_i$  and  $\mathbf{x}$  can be specified by giving its length  $r$  and the  $n$  angles  $\theta_i$ . But since  $r^2 = \mathbf{x} \cdot \mathbf{x} = r^2 \sum_{i=1}^n \cos^2 \theta_i$  we see that the  $\theta_i$  are not independent of each other. Spherical coordinates in  $n$ -dimensions show us how to pick out  $n-1$  angles  $\phi_1, \dots, \phi_{n-2}, \theta$  which are independent of each other and which, when combined with the norm  $r$ , completely describe the vector  $\mathbf{x}$  with respect to the given orthonormal basis.

**Derivation of the coordinates.** Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{x}$  be as above. Let  $\phi_1$  be the angle between  $\mathbf{x}$  and  $\mathbf{e}_1$ ,  $0 \leq \phi_1 \leq \pi$ . Then  $\xi_1 = \mathbf{x} \cdot \mathbf{e}_1 = r \cos \phi_1$  and

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + \sum_{j=2}^n \xi_j \mathbf{e}_j.$$

Now

$$r^2 = \|\mathbf{x}\|^2 = r^2 \cos^2 \phi_1 + \sum_{j=2}^n \xi_j^2 \quad \text{or} \quad \sum_{j=2}^n \xi_j^2 = r^2 \sin^2 \phi_1.$$

Setting  $\xi_j = \alpha_j r \sin \phi_1$ ,  $j=2, \dots, n$ , we have

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \sum_{j=2}^n \alpha_j \mathbf{e}_j,$$

where  $\sum_{j=2}^n \alpha_j^2 = 1$ . (If  $\phi_1$  is 0 or  $\pi$ , then  $\mathbf{x} = \pm r \mathbf{e}_1$ .)

Let  $\mathbf{u}_2 = \sum_{j=2}^n \alpha_j \mathbf{e}_j$ . The vector  $\mathbf{u}_2$  is a unit vector (that is  $\|\mathbf{u}_2\| = 1$ ) in the direction of the projection of  $\mathbf{x}$  onto the  $(n-1)$ -dimensional subspace spanned by  $\mathbf{e}_2, \dots, \mathbf{e}_n$ . If  $\phi_2$  is the angle between  $\mathbf{u}_2$  and  $\mathbf{e}_2$  then  $\cos \phi_2 = \mathbf{u}_2 \cdot \mathbf{e}_2 = \alpha_2$ ,  $0 \leq \phi_2 \leq \pi$ , and

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sum_{j=3}^n \alpha_j \mathbf{e}_j.$$

Hence,

$$1 = \|\mathbf{u}_2\|^2 = \cos^2 \phi_2 + \sum_{j=3}^n \alpha_j^2 \quad \text{or} \quad \sum_{j=3}^n \alpha_j^2 = \sin^2 \phi_2.$$

If we set  $\alpha_j = \beta_j \sin \phi_2$ ,  $j=3, \dots, n$ , then

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sin \phi_2 \sum_{j=3}^n \beta_j \mathbf{e}_j,$$

where  $\sum_{j=3}^n \beta_j^2 = 1$ . Thus

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \cos \phi_2 \mathbf{e}_2 + r \sin \phi_1 \sin \phi_2 \sum_{j=3}^n \beta_j \mathbf{e}_j.$$

In general, let  $\mathbf{u}_j$  be the unit vector in the direction of the projection of  $\mathbf{x}$  onto the space spanned by  $\mathbf{e}_j, \mathbf{e}_{j+1}, \dots, \mathbf{e}_n, j=2, \dots, n-1$  and let  $\phi_{j-1}$  be the angle between  $\mathbf{u}_j$  and  $\mathbf{e}_j, 0 \leq \phi_j \leq \pi, j=2, \dots, n-1$ . Then

$$\mathbf{x} = \sum_{j=1}^{n-2} r \left( \prod_{k=1}^{j-1} \sin \phi_k \right) \cos \phi_j \mathbf{e}_j + r \left( \prod_{k=1}^{n-2} \sin \phi_k \right) \mathbf{u}_{n-1}.$$

Now  $\mathbf{u}_{n-1} = \delta_{n-1} \mathbf{e}_{n-1} + \delta_n \mathbf{e}_n$ , where  $1 = \|\mathbf{u}_{n-1}\|^2 = \delta_{n-1}^2 + \delta_n^2$ . If now we define an angle  $\theta$  by  $\cos \theta = \delta_n, \sin \theta = \delta_{n-1}$ , we see that  $0 \leq \theta \leq \pi$  will not suffice since  $\delta_{n-1}$  can be negative and  $\sin \alpha \geq 0$  for  $0 \leq \theta \leq \pi$ . In order to include all possible combinations of  $(\delta_{n-1}, \delta_n)$  we must have  $0 \leq \theta < 2\pi$ .

Thus if  $\mathbf{e}_1, \dots, \mathbf{e}_n$  is a given orthonormal basis in  $E_n$  and  $\mathbf{x}$  is a vector of norm  $r$  with components  $\xi_j$  with respect to this basis, then

$$\begin{aligned} \xi_1 &= r \cos \phi_1, \\ \xi_j &= r \cos \phi_j \prod_{k=1}^{j-1} \sin \phi_k \quad (j = 2, \dots, n-2), \\ (*) \quad \xi_{n-1} &= r \sin \theta \prod_{k=1}^{n-2} \sin \phi_k, \\ \xi_n &= r \cos \theta \prod_{k=1}^{n-2} \sin \phi_k, \end{aligned}$$

where  $0 \leq \phi_j \leq \pi, j=1, \dots, n-2; 0 \leq \theta < 2\pi; 0 \leq r < \infty$ .

**Application to integration.** Let  $f(\xi_1, \dots, \xi_n)$  be a continuous real-valued function defined in  $E_n$  which may be written in the form

$$f(\xi_1, \dots, \xi_n) = g(\alpha_1 \xi_1 + \dots + \alpha_n \xi_n, \xi_1^2 + \dots + \xi_n^2),$$

where the  $\alpha_i$  are constants independent of the  $\xi$ 's. We wish to compute the integral of  $f$  over the surface of the  $n$ -dimensional sphere of radius  $r$  with the origin as center. If  $\mathbf{x}$  is the vector with coordinates  $\xi_j$  and  $\mathbf{a}$  the vector with coordinates  $\alpha_j$  (these coordinates being with respect to some given orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ ) then

$$\int_{\xi_1^2 + \dots + \xi_n^2 = r^2} f(\xi_1, \dots, \xi_n) dS = \int_{\|\mathbf{x}\|=r} g(\mathbf{a} \cdot \mathbf{x}, \|\mathbf{x}\|^2) dS,$$

where  $dS$  is the surface differential.

Let  $\mathbf{a}_1 = \mathbf{a}/\|\mathbf{a}\|$ \* and choose vectors  $\mathbf{a}_2, \dots, \mathbf{a}_n$  to complete an orthonormal basis in  $E_n$ . Let the coordinates of  $\mathbf{x}$  with respect to the basis  $\mathbf{a}_1, \dots, \mathbf{a}_n$  be  $\zeta_1, \dots, \zeta_n$ . Then  $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1$ . Make the spherical coordinate transformation

\* If  $\mathbf{a} = \mathbf{0}$ , then  $\mathbf{a}_1$  may be any unit vector.

given by (\*) with  $\xi_j$  replaced by  $\zeta_j, j=1, \dots, n$ .

The Jacobian of the transformation is

$$J = r^{n-1} \prod_{k=1}^{n-2} \sin^k \phi_{n-1-k}.$$

Also,  $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1 = r \cos \phi_1$ . Thus the integral becomes

$$\begin{aligned} & 2\pi r^{n-1} \left[ \prod_{k=1}^{n-3} \int_0^\pi \sin^k \phi_{n-1-k} d\phi_{n-1-k} \right] \int_0^\pi g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1 \\ &= \frac{2r^{n-1} \pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^\pi g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1. \end{aligned}$$

Thus we have reduced the integral over the surface of an  $n$ -dimensional sphere to a single integral on the real line. In particular, if  $f \equiv 1$ , we obtain  $[2\pi^{n/2}/\Gamma(n/2)]r^{n-1}$  for the surface area of an  $n$ -dimensional sphere of radius  $r$  and, integrating from 0 to  $r$ , we obtain  $[2\pi^{n/2}/(n\Gamma(n/2))]r^n$  for the volume of the sphere.

**MATRIX INTEGRATION OF  $x^k \exp(-\beta^2 x^2)$**

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Let  $V$  be the vector space of finite linear combinations of  $x^k \exp(-\beta^2 x^2)$ , fixed  $\beta, k=0, 1, \dots$ , with basis  $\{x^k \exp(-\beta^2 x^2)\}$ . Let  $D$  be a linear transformation on  $V$  which differentiates a vector belonging to  $V$ .

Since  $(x^k \exp(-\beta^2 x^2))D = kx^{k-1} \exp(-\beta^2 x^2) - 2\beta^2 x^{k+1} \exp(-\beta^2 x^2)$ , the matrix of  $D$  is

$$\begin{bmatrix} 0 & -2\beta^2 & \cdot & \cdot & \cdots & 0 & \cdots \\ 1 & 0 & -2\beta^2 & \cdot & \cdots & \cdot & \cdots \\ \cdot & 2 & 0 & -2\beta^2 & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ \cdot & \cdot & 0 & k & 0 & -2\beta^2 & \cdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

$V$  is closed under  $D$  and the kernel of  $D$  consists of the zero vector alone. The calculation of  $D^{-1}$  may be carried out algebraically, giving an interesting equation for  $\int x^k \exp(-\beta^2 x^2) dx$ .

Because of the nature of  $D, D^{-1}$  may be calculated in four independent steps depending on whether  $k$  and  $j$  are even or odd, where  $\|D^{-1}\| = \|a_{kj}\|$ . Using  $DD^{-1} = I$ , we obtain the following expressions for  $a_{kj}$ .

(1)  $j$  odd,  $k$  odd:  $a_{kj} = 0;$