The formula
\[ \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}, \]
sometimes called a *Dirichlet integral* has drawn lots of attention. G.H. Hardy [2] has a note ranking various proofs of this formula. Later [3] he reconsidered his ranking and added a proof discovered by A. C. Dixon [1]. Hardy didn’t know how to rank Dixon’s proof using his criterion and assigned it a low ranking. I will assign it a high ranking. I like it a lot. This note will present Dixon’s proof.

The convergence of \( \int_0^\infty \frac{\sin x}{x} \, dx \) can be proved using the mean value theorem for integrals as stated in problem #7, section 4.2 of Folland (it’s also true if \( \phi \) is decreasing):

\[
\left| \int_{B_1}^{B_2} \frac{\sin x}{x} \, dx \right| \leq \frac{1}{B_1} \left| \int_{B_1}^{c} \sin x \, dx \right| + \frac{1}{B_2} \left| \int_{c}^{B_2} \sin x \, dx \right| \leq 2\left(\frac{1}{B_1} + \frac{1}{B_2}\right) \to 0.
\]

These are the steps that I will verify:

1. let
   \[ u_n = \int_{\pi}^{\frac{\pi}{2}} \sin 2nx \cot x \, dx, \quad \text{(1)} \]
   \[ v_n = \int_{\pi}^{\frac{\pi}{2}} \frac{\sin 2nx}{x} \, dx. \quad \text{(2)} \]

2. Using elementary arguments We will prove
   (a) \[ \lim_{n \to \infty} v_n = \int_0^\infty \frac{\sin x}{x} \, dx. \]
   (b) \[ u_n = \frac{\pi}{2}. \]
   (c) \[ \lim_{n \to \infty} (u_n - v_n) = 0. \]

**Proof.** (a) Let \( t = 2nx \) and we see that \( v_n = \int_0^{n\pi} \frac{\sin t}{t} \, dt \to \int_0^\infty \frac{\sin t}{t} \, dt. \)

(b) We use some trigonometry. We will prove (b) by induction on \( n \). We first notice

\[ u_1 = \int_0^{\frac{\pi}{2}} \sin 2x \cot x \, dx = \int_0^{\frac{\pi}{2}} 2\cos^2 x \, dx = \int_0^{\frac{\pi}{2}} (\cos 2x + 1) \, dx = \frac{\pi}{2}. \]
The inductive step uses the following identities:

\[
\sin(2n+2)x - \sin 2nx = 2 \cos(2n+1)x \sin x,
\]

\[
2 \cos(2n+1)x \cos x = \cos(2n+2)x + \cos(2nx),
\]

which are proved using

\[
\sin(2n+1 \pm 1)x = \pm \cos(2n+1)x \sin x + \sin(2n+1)x \cos x,
\]

\[
\cos(2n+1 \pm 1)x = \cos(2n+1)x \cos x \mp \sin(2n+1)x \sin x,
\]

to show that

\[
u_{n+1} = u_n = \frac{\pi}{2}.
\]

(c) Finally we notice that \( \frac{1}{x} - \cot x \) can be defined to be \( C^1 \) at \( x = 0 \) and hence we can integrate by parts

\[
v_n - u_n = \int_0^\frac{\pi}{2} (\frac{1}{x} - \cot x) \sin 2nx \, dx = \left[ -\frac{\cos 2nx}{2n} (\frac{1}{x} - \cot x) \right]_0^\frac{\pi}{2} - \int_0^\frac{\pi}{2} \frac{\cos 2nx}{2n} (\frac{1}{x^2} - \csc^2 x) \, dx.
\]

Each of these terms goes to 0 as \( n \to \infty \). This result also follows from the Riemann-Lebesgue lemma.

I’ll include here a proof in the spirit of Dixon’s proof, although it is not his proof. I’ll make use of

**Theorem 1.** (Riemann-Lebesgue Lemma). Suppose \( f \) is Riemann integrable on \( [a, b] \). Then

\[
\lim_{n \to \infty} \int_a^b f(x) \sin nx \, dx = 0.
\]

In this statement \( \sin nx \) can be replaced by \( \cos nx \) or \( e^{\pm inx} \).

We next state a useful identity

**Proposition 1.**

\[
1 + 2 \cos 2y + 2 \cos 4y + \cdots + 2 \cos 2ny = \frac{\sin(2n+1)y}{\sin y}.
\]

This follows from the following string of identities:

\[
e^{-inx} + e^{-i(n-1)x} + \cdots + 1 + e^{ix} + \cdots + e^{inx} = e^{-inx} \frac{e^{i(n+1/2)x} - e^{-i(n+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(n + 1/2)x}{\sin x/2}.
\]

From the proposition it follows that

\[
\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)y}{\sin y} \, dy = \frac{\pi}{2}.
\]

It is also true that

\[
\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)y}{y} \, dy = \int_0^{\frac{12n+1}{2}} \frac{\sin \frac{t}{2}}{t} \, dt \to \int_0^\infty \frac{\sin \frac{t}{2}}{t} \, dt.
\]

The Riemann-Lebesgue lemma implies

\[
\int_0^{\frac{\pi}{2}} (\frac{1}{y} - \frac{1}{\sin y}) \sin((2n+1)y) \, dy, \to 0.
\]

and that concludes the proof.
References

[1] Dixon, A. C., Proof That $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, *The Mathematical Gazette*, Vol. 6, No. 96 (Jan., 1912), pp. 223-224.
