Kernels and Differential Equations

January 9, 2012

Suppose w is a solution of the following initial value problem for a linear differential equation of n^{th} order with constant coefficients:

$$Lw = w^{(n)}(x) + a_{n-1}w^{(n-1)}(x) + \cdots + a_1 w'(x) + a_0 w(x) = 0 \tag{1}$$
$$w(0) = 0, \tag{2}$$
$$w'(0) = 0, \tag{3}$$
$$\ldots, \tag{4}$$
$$w^{(n-2)}(0) = 0, \tag{5}$$
$$w^{(n-1)}(0) = 1. \tag{6}$$

Let $k(x,y) = w(x-y)$, let g be continuous, and set

$$h(x) = \int_0^x k(x,y)g(y)dy = \int_0^x w(x-y)g(y)dy.$$

Let's compute:

$$h(0) = 0, \tag{7}$$
$$h'(x) = w(0)g(x) + \int_0^x w'(x-y)g(y)dy, \tag{8}$$
$$h'(x) = \int_0^x w'(x-y)g(y)dy, \tag{9}$$
$$h'(0) = 0, \tag{10}$$
$$h''(x) = w'(0)g(x) + \int_0^x w''(x-y)g(y)dy, \tag{11}$$
$$h''(x) = \int_0^x w''(x-y)g(y)dy, \tag{12}$$
$$h''(0) = 0, \tag{13}$$
$$\ldots, \tag{14}$$
$$h^{(n-1)}(x) = \int_0^x w^{(n-1)}(x-y)g(y)dy, \tag{15}$$
$$h^{(n-1)}(0) = 0, \tag{16}$$
$$h^{(n)}(x) = w^{(n-1)}(0)g(x) + \int_0^x w^{(n)}(x-y)g(y)dy, \tag{17}$$
$$h^{(n)}(x) = g(x) + \int_0^x w^{(n)}(x-y)g(y)dy, \tag{18}$$
$$Lh(x) = g(x) + \int_0^x Lw(x-y)g(y)dy, \tag{19}$$
$$Lh(x) = g(x). \tag{20}$$

The last line is true because $Lw = 0$. The other lines are true because it is legal to differentiate under the integral sign and because of the chain rule, the fundamental theorem of calculus, and equations 1-6.

We have found a solution, represented by an integral of

$$Lh(x) = g(x), \tag{21}$$
$$h(0) = 0, \tag{22}$$
$$h'(0) = 0, \tag{23}$$
$$\ldots, \tag{24}$$
$$h^{(n-1)}(0) = 0. \tag{25}$$

It is

$$h(x) = \int_0^x w(x-y)g(y)dy.$$