As this ratio tends to zero, when $\alpha > 1$, but $\to +\infty$, when $\alpha \leqq 1$, *Ermakoff*'s test therefore provides the known conditions for convergence and divergence of these series, as asserted[28].

2. We may of course make use of other functions instead of e^x. If $\varphi(x)$ is any monotone increasing positive function, everywhere differentiable, for which $\varphi(x) > x$ always, the series Σa_n will converge or diverge according as we have

$$\frac{\varphi'(x) f(\varphi(x))}{f(x)} \begin{cases} \leqq \vartheta < 1 \\ \geqq 1 \end{cases}$$

for all sufficiently large x's

With *Ermakoff*'s test and *Cauchy*'s integral test, we have command over the most important tests for our present series.

§ 41. General remarks on the theory of the convergence and divergence of series of positive terms.

Practically the whole of the 19th century was required to establish the convergence tests set forth in the preceding sections and to elucidate their meaning. It was not till the end of that century, and in particular by *Pringsheim*'s investigations, that the fundamental questions were brought to a satisfactory conclusion. By these researches, which covered an extremely extensive field, a series of questions were also solved, which were only timidly approached before his time, although now they appear to us so simple and transparent that it seems almost inconceivable that they should have ever presented any difficulty[29], still more so, that they should have been answered in a completely erroneous manner. How great a distance had to be traversed before this point could be reached is clear if we reflect that *Euler* never troubled himself at all about questions of convergence; when a series occurred, he would attribute to it, without any hesitation, the value of the expression which gave rise to the series[30]. *Lagrange* in 1770[31] was still of the opinion that a series represents a definite value, provided only that its terms decrease to 0[32]. To refute the latter

[28] This also holds for $p = 0$, if we interpret $\log_{-1} x$ to mean e^x.

[29] As a curiosity, we may mention that, as late as 1885 and 1889, several memoirs were published with the object of demonstrating the existence of convergent series $\sum c_n$ for which $\frac{c_{n+1}}{c_n}$ did not tend to a limit! (Cf. **159**, 3.)

[30] Thus in all seriousness he deduced from $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, that

$$\frac{1}{2} = 1 - 1 + 1 - 1 + - \cdots$$

and

$$\frac{1}{3} = 1 - 2 + 2^2 - 2^3 + - \cdots.$$

Cf the first few paragraphs of § 59.

[31] V. *Œuvres*, Vol. 3, p. 61.

[32] In this, however, some traces of a sense for convergence may be seen.

assumption expressly by referring to the fact (at that time already well known) of the divergence of $\sum \frac{1}{n}$, appears to us at present superfluous, and many other presumptions and attempts at proof current in previous times are in the same case. Their interest is therefore for the most part historical. A few of the questions raised, however, whether answered in the affirmative or negative, remain of sufficient interest for us to give a rapid account of them. A considerable proportion of these are indeed of a type to which anyone who occupies himself much with series is naturally led.

The source of all the questions which we propose to discuss resides in the inadequacy of the criteria. Those which are necessary *and* sufficient for convergence (the main criterion **81**) are of so general a nature, that in particular cases the convergence can only rarely be ascertained by their means. All our remaining tests (comparison tests or transformations of comparison tests) were *sufficient* criteria only, and they only enabled us to recognise as convergent series which converge at least as rapidly as the comparison series employed. The question at once arises:

1. *Does a series exist which converges less rapidly than any other?* **178.**
This question is already answered, in the negative, by the theorem **175**, 4. In fact, when $\sum c_n$ converges, so does $\sum c_n' = \sum \frac{c_n}{r_{n-1}^{\frac{1}{2}}}$, though, obviously, less rapidly than $\sum c_n$, as $c_n : c_n' = r_{n-1}^{\frac{1}{2}} \to 0$.

The question is answered almost more simply by *J. Hadamard*[33], who takes the series $\sum c_n' \equiv \sum (\sqrt{r_{n-1}} - \sqrt{r_n})$. Since $c_n = r_{n-1} - r_n$, the ratio $c_n : c_n' = \sqrt{r_{n-1}} + \sqrt{r_n} \to 0$. The accented series converges less rapidly than the unaccented series.

The next question is equally easy to solve:

2. *Does a series exist which diverges less rapidly than any other?* Here again, the theorem of *Abel-Dini* **173** shows us that when $\sum d_n$ diverges, so does $\sum d_n' = \sum \frac{d_n}{D_n}$, and hence the answer has to be in the negative. In fact as $d_n : d_n' = D_n \to +\infty$, the theorem provides, for each given divergent series, another whose divergence is not so rapid.

These circumstances, together with our preliminary remarks, show that

3. *No comparison test can be effective with all series.*

Closely connected with this, we have the following question, raised and also answered, by *Abel*[34]:

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[33] Acta mathematica, Vol. 18, p. 319. 1894.
[34] J. f. d. reine u. angew. Math., Vol. 3, p. 80. 1828

4. *Can we find positive numbers p_n, such that, simultaneously,*

$$\left.\begin{array}{l}\text{a) } p_n a_n \to 0 \\ \text{b) } p_n a_n \geqq \alpha > 0\end{array}\right\} \textit{ are sufficient conditions for } \left\{\begin{array}{l}\textit{convergence} \\ \textit{divergence}\end{array}\right\}$$

of every possible series of positive terms?

It again follows from the theorem of *Abel-Dini* that this is not the case. In fact, if we put $a_n = \frac{\alpha}{p_n}$, $\alpha > 0$, the series Σa_n necessarily diverges, and hence so does $\Sigma a_n' \equiv \sum \frac{a_n}{s_n}$, where $s_n = a_1 + \cdots + a_n$. But, for the latter, $p_n a_n' = \frac{\alpha}{s_n} \to 0$.

The object of the comparison tests was, to some extent, the construction of the *widest* possible conditions *sufficient* for the determination of the convergence or divergence of a series. Conversely, it might be required to construct the *narrowest* possible conditions *necessary* for the convergence or divergence of a series. The only information we have so far gathered on this subject is that $a_n \to 0$ is necessary for convergence. It will at once occur to us to ask:

5. *Must the terms a_n of a convergent series tend to zero with any particular rapidity?* It was shown by *Pringsheim*[35] that this is not the case. However slowly the numbers p_n may tend to $+\infty$, we can invariably construct convergent series Σc_n for which

$$\overline{\lim}\, p_n c_n = +\infty.$$

Indeed *every* convergent series $\Sigma c_n'$, by a suitable rearrangement, will produce a series Σc_n to support this statement[36].

Proof. We assume given the numbers p_n, increasing to $+\infty$, and the convergent series $\Sigma c_n'$. Let us choose the indices $n_1, n_2, \ldots, n_\nu, \ldots$ *odd* and such that

$$\frac{1}{p_{n_\nu}} < \frac{c'_{2\nu-1}}{\nu} \qquad (\nu = 1, 2, \ldots)$$

and let us write $c_{n_\nu} = c'_{2\nu-1}$, filling in the remaining c_n's with the terms $c_2', c_4', \ldots$ in their original order. The series Σc_n is obviously a rearrangement of $\Sigma c_n'$. But

$$p_n c_n > \nu$$

whenever n becomes equal to one of the indices n_ν. Accordingly, as asserted,

$$\overline{\lim}\, p_n c_n = +\infty.$$

The underlying fact in this connection is simply that the behaviour of a *sequence* of the form $(p_n c_n)$ bears no essential relation to that of

[35] Math. Annalen, Vol. 35, p 344. 1890

[36] Cf. Theorem **82**, 3, which takes into account a sort of decrease on the average of the terms a_n.

the *series* Σc_n — i. e. with the sequence of *partial sums* of this series, — since the latter, though not the former, may be fundamentally altered by a rearrangement of its terms.

6. Similarly, *no condition of the form* $\overline{\lim}\, p_n d_n > 0$ *is necessary for the divergence of* Σd_n, *however rapidly* the positive numbers p_n may increase to $+\infty$[37]. On the contrary, *every* divergent series $\Sigma d_n'$, provided its terms tend to 0, becomes, on being suitably rearranged, a series Σd_n (still divergent, of course) for which $\underline{\lim}\, p_n d_n = 0$. — The proof is easily deduced on the same lines as the preceding.

The following question goes somewhat further:

7. *Does a scale of comparison tests exist which is sufficient for all cases?* More precisely: Given a number of convergent series

$$\Sigma c_n^{(1)}, \quad \Sigma c_n^{(2)}, \ldots, \Sigma c_n^{(k)}, \ldots$$

each of which converges less rapidly than the preceding, with e. g.

$$\frac{c_n^{(k+1)}}{c_n^{(k)}} \to +\infty, \qquad \text{for fixed } k.$$

(The logarithmic scale affords an example of such series.) *Is it possible to construct a series converging less rapidly than any of the given series?* The answer is in the affirmative[38]. The actual construction of such a series is indeed not difficult. With a suitable choice of the indices $n_1, n_2, \ldots, n_k, \ldots$, the series

$$c_n \equiv c_1^{(1)} + c_2^{(1)} + \cdots + c_{n_1}^{(1)} + c_{n_1+1}^{(2)} + \cdots + c_{n_2}^{(2)} + c_{n_2+1}^{(3)} + \cdots + c_{n_3}^{(3)} + c_{n_3+1}^{(4)} + \cdots$$

is itself of the kind required. We need only choose these indices so large that if we denote by $r_n^{(k)}$ the remainder, after the n^{th} term, of the series $\Sigma c_n^{(k)}$,

for every $n \geqq n_1$, we have $r_n^{(2)} < \frac{1}{2}$ with $c_n^{(2)} > 2\, c_n^{(1)}$

" " $n \geqq n_2 > n_1$, " " $r_n^{(3)} < \frac{1}{2^2}$ " $c_n^{(3)} > 2\, c_n^{(2)}$

. . . .

" " $n \geqq n_k > n_{k-1}$ " " $r_n^{(k+1)} < \frac{1}{2^k}$ " $c_n^{(k+1)} > 2\, c_n^{(k)}$

. . . .

The series Σc_n is certainly convergent, for each successive portion of it belonging to one of the series $\Sigma c_n^{(k)}$ is certainly less than the

[37] *Pringsheim*, loc. cit. p. 357

[38] For the logarithmic scale, this was shewn by *P. du Bois-Reymond* (J. f. d. reine u. angew. Math., Vol. 76, p. 88. 1873). The above extended solution is due to *J. Hadamard* (Acta math., Vol. 18, p. 325. 1894).

remainder of this series, starting with the same initial term, i. e $< \frac{1}{2^k}$ $(k = 2, 3, \ldots)$. On the other hand, for *every fixed* k,

$$\frac{c_n}{c_n^{(k)}} \to +\infty;$$

in fact for $n > n_q$ $(q > k)$ we have obviously $\frac{c_n}{c_n^{(k)}} > 2^{q-k}$. This proves all that was required. — In particular, there are series converging more slowly than all the series of our logarithmic scale[39].

8. We may show, quite as simply, that, given a number of divergent series $\Sigma d_n^{(k)}$, $k = 1, 2, \ldots$, each diverging less rapidly than the preceding, with, specifically, $d_n^{(k+1)} \div d_n^{(k)} \to 0$, say, there are always divergent series Σd_n diverging less rapidly than *every one* of the series $\Sigma d_n^{(k)}$.

All the above remarks bring us near to the question whether and to what extent the terms of convergent series are fundamentally distinguishable from those of divergent series. In consequence of 7. and 8., we shall no longer be surprised at the observation of *Stieltjes*:

9. *Denoting by* $(\varepsilon_1, \varepsilon_2, \ldots)$ *an arbitrary monotone descending sequence with limit* 0, *a convergent series* Σc_n *and a divergent series* Σd_n *can always be specified, such that* $c_n = \varepsilon_n d_n$. — In fact, if $\varepsilon_n \to 0$ monotonely, $p_n = \frac{1}{\varepsilon_n} \to +\infty$ monotonely. The series

$$p_1 + (p_2 - p_1) + \cdots + (p_n - p_{n-1}) + \cdots,$$

whose partial sums are the numbers p_n, is therefore divergent. By the theorem of *Abel-Dini*, the series

$$\sum_{n=1}^{\infty} d_n \equiv \sum_{n=1}^{\infty} \frac{p_{n+1} - p_n}{p_{n+1}}$$

is also divergent. But the series $\Sigma c_n \equiv \Sigma \varepsilon_n d_n \equiv \Sigma \left(\frac{1}{p_n} - \frac{1}{p_{n+1}}\right)$ is convergent by **131**. —

The following remark is only a re-statement in other words of the above:

10. *However slowly* $p_n \to +\infty$, *there is a convergent series* Σc_n *and a divergent series* Σd_n *for which* $d_n = p_n c_n$.

In this respect, the two remarks due to *Pringsheim*, given in 5. and 6., may be formulated even more forcibly as follows:

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[39] The missing initial terms of these series may be assumed to be each replaced by unity.

11. *However rapidly $\sum c_n$ may converge, there are always divergent series,* — ***indeed divergent series with monotonely diminishing terms of limit* 0,** — *for which*

$$\underline{\lim} \frac{d_n}{c_n} = 0.$$

Thus $\sum d_n$ must have an infinite number of terms essentially *smaller* than the corresponding terms of $\sum c_n$. Conversely:

However rapidly $\sum d_n$ may diverge, provided only $d_n \to 0$, there are always convergent series $\sum c_n$ for which $\overline{\lim} \frac{c_n}{d_n} = +\infty$.

We have only to prove the former statement. Here a series $\sum d_n$ of the form

$$\sum_{n=0}^{\infty} d_n \equiv c_1 + c_1 + \cdots + c_1 + \frac{1}{2} c_{n_1} + \frac{1}{2} c_{n_1} + \cdots + \frac{1}{2} c_{n_1}$$
$$+ \frac{1}{3} c_{n_2} + \frac{1}{3} c_{n_2} + \cdots + \frac{1}{3} c_{n_2} + \frac{1}{4} c_{n_3} + \cdots$$

is of the required kind, if the increasing sequence of indices $n_1, n_2, \ldots$ be chosen suitably and the successive groups of equal terms contain respectively n_1, $(n_2 - n_1)$, $(n_3 - n_2)$, ... terms. In fact, in order that this series may diverge, it is sufficient to choose the number of terms in each group so large that their sum > 1, and in order that the sequence of terms in the series be monotone, it is sufficient to choose $n_k > n_{k-1}$ so large that $c_{n_k} < c_{n_{k-1}}$ $(k = 1, 2, \ldots; n_0 = 1)$ as is always possible, since $c_n \to 0$. As the ratio $\frac{d_n}{c_n}$ has the value $\frac{1}{k+1}$ for $n = n_k$, it follows that $\underline{\lim} \frac{d_n}{c_n} = 0$, as required.

In the preceding remarks we have considered only convergence or divergence *per se*. It might be hoped that with narrower requirements, e. g. that the terms of the series should diminish monotonely, a correspondingly greater amount of information could be obtained. Thus, as we have seen, for a convergent series $\sum c_n$ whose terms diminish monotonely, we have $n c_n \to 0$. *Can more than this be asserted?* The answer is in the negative (cf. Rem. 5):

12. *However slowly the positive numbers p_n may increase to $+\infty$, there are always convergent series of monotonely diminishing terms for which*

$$n p_n c_n$$

not only does not tend to 0, *but has* $+\infty$ *for upper limit*[40].

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[40] *Pringsheim*, loc. cit. In particular it was much discussed whether for convergent series of positive terms, diminishing monotonely, the expression $n \log n \cdot c_n$ must $\to 0$; the opinion was held by many, as late as 1860, that $n \log n \cdot c_n \to 0$ was necessary for convergence.

The *proof* is again quite easy. Choose indices $n_1 < n_2 < \cdots$ such that

$$p_{n_\nu} > 4^\nu \qquad (\nu = 1, 2, \ldots)$$

and write

$$c_1 = c_2 = \cdots = c_{n_1} = \frac{1}{n_1 \sqrt{p_{n_1}}},$$

$$c_{n_1+1} = c_{n_1+2} = \cdots = c_{n_2} = \frac{1}{n_2 \sqrt{p_{n_2}}},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$c_{n_{\nu-1}+1} = \cdots\cdots = c_{n_\nu} = \frac{1}{n_\nu \sqrt{p_{n_\nu}}},$$

$$\cdots\cdots\cdots\cdots\cdots$$

The groups of terms here indicated contribute successively less than $\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^\nu}, \ldots$ to the sum of the series Σc_n, so that this series will converge. On the other hand, for each $n = n_\nu$ we have

$$n\, p_n\, c_n = \sqrt{p_n},$$

so that, as was required,

$$\overline{\lim}\; n \cdot p_n \cdot c_n = +\infty.$$

13. These remarks may easily be multiplied and extended in all possible directions. They make it clear that it is *quite useless* to attempt to introduce anything of the nature of a *boundary between convergent and divergent series,* as was suggested by *P. du Bois-Reymond.* The notion involved is of course vague at the outset. But in whatever manner we may choose to render it precise, it will never correspond to the actual circumstances. We may illustrate this on the following lines, which obviously suggest themselves[41].

a) As long as the terms of the series Σc_n and Σd_n are subjected to no restriction (excepting that of being > 0), the ratio $\frac{c_n}{d_n}$ is capable of assuming all possible values, as besides the inevitable relation

$$\underline{\lim}\, \frac{c_n}{d_n} = 0 \quad \text{we may also have} \quad \overline{\lim}\, \frac{c_n}{d_n} = +\infty.$$

The polygonal graphs by which the two sequences (c_n) and (d_n) may be represented, in accordance with 7, 6, can therefore intersect at an indefinite number of points (which may grow more and more numerous, to an arbitrary extent).

[41] A detailed and careful discussion of all the questions belonging to the subject will be found in *Pringsheim*'s work mentioned on p. 2, and also in his writings in the Math. Ann. Vol. 35 and in the Munch. Ber. Vol. 26 (1896) and 27 (1897), to which we have repeatedly referred.

b) By our remark 11, this remains true when the two sequences (c_n) and (d_n) are *both monotone*, in which case the graphs above referred to are *both monotone descending* polygonal lines. It is therefore certainly not possible to draw a line stretching to the right, with the property that *every* sequence of type (c_n) has a graph, *no part of which lies above* the line in question, and every sequence of type (d_n) a graph, *no part of which lies below* this line, — even if the two graphs are monotone and are considered only from some point situated at a sufficiently great distance to the right.

14. Notes 11 and 12 suggest the question whether the statements there made remain unaltered if the terms of the constructed series Σc_n and Σd_n are not merely ***simply*** *monotone* as above, but ***fully*** *monotone* in the sense of p. 263. This question has been answered in the affirmative by *H. Hahn* [42].

§ 42. Systematization of the general theory of convergence.

The element of chance inherent in the theory of convergence as developed so far gave rise to various attempts to systematize the criteria from more general points of view. The first extensive attempts of this kind were made by *P. du Bois-Reymond* [43], but were by no means brought to a conclusion by him. *A. Pringsheim* [44] has been the first to accomplish this, in a manner satisfactory both from a theoretical and a practical standpoint. We propose to give a short account of the leading features of the developments due to him [45].

All the criteria set forth in these chapters have been comparison tests, and their common source is to be found in the two comparison tests of the first and second kinds, **157** and **158**. The former, namely

$$\text{(I)} \qquad a_n \leqq c_n \quad : \quad \mathfrak{C}, \qquad a_n \geqq d_n \quad : \quad \mathfrak{D},$$

is undoubtedly the simplest and most natural test imaginable; not so that of the second kind, given originally in the form

$$\text{(II)} \qquad \frac{a_{n+1}}{a_n} \leqq \frac{c_{n+1}}{c_n} \quad : \quad \mathfrak{C}, \qquad \frac{a_{n+1}}{a_n} \geqq \frac{d_{n+1}}{d_n} \quad : \quad \mathfrak{D}.$$

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[42] *H. Hahn*, Über Reihen mit monoton abnehmenden Gliedern, Monatsheft f. Math. u. Physik, Vol. 33, pp. 121—134, 1923.

[43] J. f. d. reine u. angew. Math. Vol. 76, p. 61. 1873.

[44] Math. Ann. Vol. 35, pp. 297—394. 1890.

[45] We have all the more reason for dispensing with details in this connexion, seeing *Pringsheim*'s researches have been developed by the author himself in a very complete, detailed, and readily accessible form.