Poincaré Lemma

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There are several proofs of the Poincaré lemma. This note is an exposition of a proof that appears in [1]. It is more than a proof. It is an algorithm for finding a form μ so that $\omega = d\mu$ when $d\omega = 0$ on a rectangular parallelopiped in $\mathbf{R}^n$.

We need a little information about differential forms. This discussion is brief, but should be enough to define notation. If $x_1, x_2, \ldots, x_n$ are coordinates in $\mathbf{R}^n$, differential forms are (ordered) objects such as $adx_2dx_3dx_5$ which satisfy such rules as

$$adx_2dx_3dx_5 = -adx_2dx_5dx_3.$$

In other words interchanging adjacent terms changes signs, so

$$adx_2dx_3dx_5 = adx_5dx_2dx_3,\ dx_2dx_2 = 0$$

In this notation, a is a differentiable function. The crucial operator is the *exterior derivative.* First it is defined for functions:

$$df = f_{x_1}dx_1 + f_{x_2}dx_2 + \ldots f_{x_n}dx_n,$$

then individual terms:

$$d(adx_I) = dadx_I,$$

where I is a multi-index,

$$I = (i_1, i_2, \ldots, i_k),\ dx_I = dx_{i_1}dxi_2 \ldots dx_{i_k},$$

and extended linearly. A form ω is *closed* if $d\omega = 0$ and *exact* if $\omega = d\mu$. Then since $d^2 = 0$, all exact forms are closed. The Poincaré Lemma is

Lemma 1. *On a rectangular parallelopiped,* Π, *all closed forms are exact.*

Proof. This note will present the proof for a somewhat complicated special case. The general proof goes by induction. It should be clear from the steps in the proof that the proof is constructive. The proof will be given for a closed 2-form, ω, in 4-space. Let

$$\omega = \alpha_1 + Pdxdw + Qdydw + Rdzdw,$$

where α_1 does not involve dw. On Π we can find functions p, q, r so that $p_w = P, q_w = Q, r_w = R$ (what's the reason for this?). Now let $\beta_1 = pdx + qdy + rdz$, and let $\omega_1 = \omega + d\beta_1$. Then $\omega + d\beta_1$ does not have any terms involving dw and $d\omega_1 = d\omega + d^2\beta_1 = 0$. Let $\omega_1 = Adxdy + Bdxdz + Cdydz$. Since $d\omega_1 = 0$ it follows that $B_w = 0, C_w = 0$. Now since we are on a convex domain, B and C do not depend on w (homework excercise 2.4, # 2). Let $B = b_z, C = c_z$ and $\beta_2 = bdx + cdy$. Then $\omega_2 = \omega_1 + d\beta_2 = Fdxdy$ has no terms involving dw or dz. Also $d\omega_2 = 0$. This last equation implies that $F_z = 0, F_w = 0$. Let $f_y = F$, where f, F depend only on x, y. Let $\beta_3 = fdx$ and notice that $\omega_2 + d\beta_3 = 0$. Writing this out we see

$$\omega + d\beta_1 + d\beta_2 + d\beta_3 = 0.$$

Hence

$$\omega = d(-\beta_1 - \beta_2 - \beta_3).$$

□

References

1. Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.