Helmholtz Theorem

Helmholtz published a theorem in 1858 that is now referred to as the Helmholtz Decomposition Theorem, but may have been found earlier by Stokes in 1849. I haven't looked at these papers. This note presents a statement and proof of a simple version of the theorem. We will use equation 5.33 in Folland and Theorem 5.46.

Theorem 1. *Let a be a C^2 vector field in $\mathbb{R}^3$ with compact support. Let*

$$v(x) = \frac{-1}{4\pi} \int_{R^3} \frac{a(x,y)}{|y|} dy.$$

Then

$$a(x) = grad(div(v(x))) - curl(curl(v(x)))$$

Proof. By formula 5.33

$$div(grad(v(x))) = grad(div(v(x))) - curl(curl(v(x))),$$

where $div(grad(v(x)))$ is computed component by component. By 5.46, looking at each component we see that

$$div(grad(v(x))) = a(x).$$

We have proved that it is ok to differentiate unde the integral sign. □