Bump Functions

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This note describes how to make a C^∞ bump function with compact support. The exposition is taken from Jack Lee's book, *Introduction to Smooth Manifolds*. We are already familiar with the function $f(x) = e^{-1/x^2}$, if $x \neq 0$; 0, if $x = 0$. See *Folland* exercise 9, §2.1. Now define a new C^∞ function $h(x)$ by

$$h(x) = \begin{cases} e^{-1/x^2}, \text{ if } x > 0, \\ 0, \text{ if } x \leq 0. \end{cases}$$

The same argument used in exercise 9 can be used to prove that h is C^∞. We define a C^∞ function g by

$$g(x) = \frac{h(2-x)}{h(2-x) + h(x-1)}.$$

Then

$$\begin{aligned} g(x) =& 1, \text{ if } x < 1, \\ g(x) =& 0, \text{ if } x > 2, \\ 0 \leq g(x) \leq& 1, \text{ if } 1 \leq x \leq 2. \end{aligned}$$

Finally define

$$b(x) = g(|x|).$$

Then $b(x) = 0$ if $|x| > 2$, $b(x) = 1$ if $|x| < 1$, and $0 \leq b(x) \leq 1$ if $1 \leq |x| \leq 2$. Also b is C^∞, since it is clearly C^∞ if $|x| < 1$ or $|x| > 2$; and since $|x|$ is C^∞ when $x \neq 0$, b is the composition of C^∞ functions for $1 \leq |x| \leq 2$.

If we use a linear change of coordinates we can create a C^∞ function

$$b_{a,b}(x) = b(-2 + 4\frac{x-a}{b-a}),$$

such that $b_{a,b}(x) > 0$ in (a, b) and $b_{a,b} = 0$ if $x \notin [a, b]$.