Math 335 Sample Problems

One notebook sized page of notes (*one side*)will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5-4.7 and 5.2-5.8. There may be homework problems on the test. The midterm is on Monday, January 29.

1. Suppose f is continuous on $[0, \infty)$ and $|xf(x)| < 1$ for $x \geq 1$. Prove or give a counterexample to the statement that $\int_1^\infty f(x)dx$ converges.

2. Let C be the curve of intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ oriented in the counterclockwise direction when viewed from a point high on the z-axis. Use Stokes' theorem to compute the value of $\displaystyle\int_C (xz + 1)dx + (yz + 2x)dy$.

3. Let
$$\phi(x) = \int_0^\pi \cos(x \sin t)dt.$$
Prove that
$$x\phi''(x) + \phi'(x) + x\phi(x) = 0.$$

4. (a) Prove that $\displaystyle\int_C \frac{-ydx + xdy}{x^2 + y^2}$ is not independent of path on $\mathbf{R}^2 - \mathbf{0}$.

   (b) Prove that $\displaystyle\int_C \frac{xdx + ydy}{x^2 + y^2}$ is independent of path on $\mathbf{R}^2 - \mathbf{0}$. Find a function $f(x, y)$ on $\mathbf{R}^2 - \mathbf{0}$ so that $\nabla f = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$.

5. A surface S containing the point $(1, 2, 3)$ has a tangent plane whose equation at each point (a, b, c) of S is
$$(a + c)(x - a) - (b + c)(y - b) + (a - b)(z - c) = 0.$$
Find the equation of the surface in the form $f(x, y, z) = 0$.

6. Prove that $\displaystyle\int_0^\infty \cos x^2 dx$ converges, but not absolutely.

7. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

   (a)
$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

(b)
$$\int_0^{\pi} \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

(c)
$$\int_1^{\infty} \frac{\sin(1/x)}{x} dx$$

8. Let f and g be integrable on $[a, b]$ for every $b > a$.

   (a) Prove that
   $$(\int_a^b |fg|)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

   (b) Prove that if $\int_a^\infty f^2$ and $\int_a^\infty g^2$ converge then $\int_a^\infty fg$ converges absolutely.

9. (a) Compute $\displaystyle\int_{x^2+y^2=1} \frac{-ydx + xdy}{x^2 + y^2}$.

   (b) Using part (a) and Green's theorem, compute $\displaystyle\int_{\frac{x^2}{4}+\frac{y^2}{9}=1} \frac{-ydx + xdy}{x^2 + y^2}$.

10. Let S be the surface (torus) obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ around the z-axis. Compute the integral $\displaystyle\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$.

11. Let $w(x)$ satisfy $w''(x) + w(x) = 0, w(0) = 0, w'(0) = 1$. Let $f(x) = \displaystyle\int_0^x (w(x - y))h(y)dy$. Prove that
    $$f''(x) + f(x) = h(x), f(0) = 0, f'(0) = 0.$$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and assume $f(\theta) > 0$. Use Green's theorem to prove that area of the region S, defined in polar coordiates by the inequalities
    $$\alpha \leq \theta \leq \beta, r \leq f(\theta),$$
    is given by
    $$A(S) = \frac{1}{2} \int_\alpha^\beta f^2(\theta) d\theta.$$

13. We have covered the following:

    (a) Green's theorem
    (b) Surface area
    (c) Divergence theorem
    (d) Stokes' theorem

(e) Integrating vector derivatives

(f) Integrals dependent on a parameter

(g) Improper single and multiple integrals

14. There may be homework problems or example problems from the text on the midterm.