Series Stuff

Theorem 1 (Root Test)**.** *Cauchy, 1821. Let* $\sum a_n$ *be a series with* $a_n \geq 0$*. Let* $r = \overline{\lim_{n\to\infty}}\, a_n^{1/n}$*.*

1. *If* $r < 1$ *the series converges.*

2. *If* $r > 1$ *the series diverges.*

Proof. 1. If $r < 1$ there is a number $c < 1$ so that if N is large enough $a_n \leq c^n$ so the series converges by comparison to the geometric series $\sum c^n$.

2. If $r > 1$ then for infinitely many indices n, $a_n > 1$ so the terms $a_n \not\to 0$. Hence $\sum a_n$ does not converge.

□

Corollary 1. *Let* $\sum a_n$ *be a series. Let* $r = \overline{\lim_{n\to\infty}}\, |a_n|^{1/n}$*.*

1. *If* $r < 1$ *the series converges absolutely.*

2. *If* $r > 1$ *the series diverges.*

Proof. If $r < 1$ the series $\sum |a_n|$ converges. If $r > 1$, by the proof of the theorem, for infinitely many indices n, $|a_n| > 1$ so the terms a_n do not go to 0. □

Theorem 2. *Let* $a_n > 0$*. Then*

$$\underline{\lim_{n\to\infty}}\, \frac{a_{n+1}}{a_n} \leq \underline{\lim_{n\to\infty}}\, a_n^{1/n} \leq \overline{\lim_{n\to\infty}}\, a_n^{1/n} \leq \overline{\lim_{n\to\infty}}\, \frac{a_{n+1}}{a_n}.$$

Hence if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ *then* $\lim_{n\to\infty} a_n^{1/n} = L$*.*

Proof. The middle inequality is obvious. I'll prove the first inequality. Let $c = \underline{\lim_{n\to\infty}}\, \frac{a_{n+1}}{a_n}$. If $c = 0$ there is nothing to prove so assume $c > 0$. Let d be any number with $c > d > 0$. Then there is N so that if $n \geq N$

$$\frac{a_{n+1}}{a_n} > d.$$

Multiplying these inequalities and taking n^{th} roots we get

$$\frac{a_n}{a_N} > d^{n-N} \tag{1}$$

$$a_n^{1/n} > d\left(a_N d^{-N}\right)^{1/n}. \tag{2}$$

Now let $n \to \infty$ to get

$$\underline{\lim_{n\to\infty}}\, a_n^{1/n} \geq d.$$

But d was any number less than c. So

$$\underline{\lim}_{n\to\infty} a_n^{1/n} \geq c = \underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

□

Theorem 3 (Ratio Test)**.** *D'Alembert, 1768. Let* $a_n > 0$*.*

1. *If* $\overline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, $\sum a_n$ *converges.*
2. *If* $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, $\sum a_n$ *diverges.*

Proof. Use Theorems 1 and 2. □

These results prove that if the ratio test implies converence or divergence then the root test would also have given the same answer (root test is better than the ratio test.)

Theorem 4 (Condensation Test)**.** *Cauchy, 1821. Let* $a_n \geq a_{n+1} \geq 0$*. Then*

$$\sum a_n < \infty \iff \sum 2^n a_{2^n} < \infty.$$

Proof. I'll prove one direction. By comparison

$$\frac{1}{2}(2^0 a_1 + 2^1 a_2 + 2^2 a_4 + 2^3 a_8 + \dots) \leq \frac{1}{2} a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \dots,$$

implies that if $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \dots < \infty$ then $\sum 2^n a_{2^n} < \infty$.

□

Corollary 2.

$$\sum \frac{1}{n}$$

diverges.

Proof.

$$\sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty.$$

□

Theorem 5. *e is irrational.*

Proof. It's easy to show that

$$\begin{aligned} 0 < e - (1 + 1 + 1/2! + 1/3! + \dots + 1/n!) &= 1/(n+1)! + 1/(n+2)! + \dots \\ &< \frac{1}{(n+1)!}(1 + 1/(n+1) + 1/(n+1)^2 + 1/(n+1)^3 + \dots \\ &= \frac{1}{n \cdot n!}. \end{aligned}$$

Suppose $e = m/n$. Now multiply by $n!$ to get

$$0 < m \cdot (n-1)! - (n! + \dots + 1) < 1/n.$$

Since the middle term is an integer this is a contradiction. □