

Math 335 Sample Problems

One notebook sized page of notes (*one side*) will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5-4.7, 5.3-5.8, and 6.1-6.3. The midterm is on Monday, February 2.

1. Let C be the curve of intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ oriented in the counterclockwise direction when viewed from a point high on the z -axis. Use Stokes' theorem to compute the value of $\int_C (xz + 1)dx + (yz + 2x)dy$.

2. (a) Prove that $\int_C \frac{-ydx + xdy}{x^2 + y^2}$ is not independent of path on $\mathbf{R}^2 - \mathbf{0}$.

- (b) Prove that $\int_C \frac{xdx + ydy}{x^2 + y^2}$ is independent of path on $\mathbf{R}^2 - \mathbf{0}$. Find a function $f(x, y)$ on $\mathbf{R}^2 - \mathbf{0}$ so that $\nabla f = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$.

3. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\sum_1^\infty a_n$ diverges.

4. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

5. Let x_n be a convergent sequence and let $c = \lim_{n \rightarrow \infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and

$$\sum_1^\infty a_n = x_1 + x_2 + \dots + x_p - pc.$$

6. Suppose $a_n > 0$, $b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

7. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n \in S} \frac{1}{n}$ converges.

8. Prove that $\int_0^{\infty} \cos x^2 dx$ converges, but not absolutely.

9. Let $a = \lim_{n \rightarrow \infty} a_n$. Prove that $\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = a$.

10. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

(a)

$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

(b)

$$\int_0^{\pi} \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

(c)

$$\int_1^{\infty} \frac{\sin(1/x)}{x} dx$$

11. Let f and g be integrable on $[a, b]$ for every $b > a$.

(a) Prove that

$$\left(\int_a^b |fg| \right)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

(b) Prove that if $\int_a^{\infty} f^2$ and $\int_a^{\infty} g^2$ converge then $\int_a^{\infty} fg$ converges absolutely.

12. (a) Suppose $\sum_1^{\infty} a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n \rightarrow \infty} (a_n + a_{n+1} + \cdots + a_{n+p}) = 0$.

(b) Suppose $\lim_{n \rightarrow \infty} (a_n + a_{n+1} + \cdots + a_{n+p}) = 0$ for every p . Does $\sum_1^{\infty} a_n$ converge?

13. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\sum_1^{\infty} a_n$ converges if and only if $\sum_0^{\infty} a_{3n}$ converges.

14. Let S be the surface (torus) obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z -axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$.

15. Let $w(x)$ satisfy $w''(x) + w(x) = 0$, $w(0) = 0$, $w'(0) = 1$. Let $f(x) = \int_0^x (w(x-y))h(y)dy$. Prove that

$$f''(x) + f(x) = h(x), f(0) = 0, f'(0) = 0.$$

16. Let f be a continuously differentiable function on $[a, b]$ and suppose $f(x) > 0$ if $x \in [a, b]$. Let S be the surface in \mathbb{R}^3 obtained by rotating the graph of f around the x -axis in \mathbb{R}^3 . Prove that the area of S is given by $2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx$.

17. In the notation of the previous problem, let $f(x) = \frac{1}{x}$, $a = 1$, $b = t$. Prove that the area of the surface of revolution goes to $+\infty$ as $t \rightarrow +\infty$, but the volume stays bounded (goes to π). (You can't paint this surface, but you can fill it with paint).

18. We have covered the following:

- (a) Divergence theorem
- (b) Stokes' theorem
- (c) Integrating vector derivatives
- (d) Integrals dependent on a parameter
- (e) Improper single and multiple integrals
- (f) Convergence and divergence of a series
- (g) Comparison test
- (h) Integral test
- (i) Cauchy condensation test
- (j) Root test and ratio test
- (k) Absolute and conditional convergence of a series

19. There may be homework problems or example problems from the text on the midterm.