Abel Theorems

This document will prove two theorems with the name Abel attached to them. Abel proved the result on series in an 1826 paper. I can find no reference to a paper of Abel in which he proved the result on Laplace transforms.

Theorem 1. *[1] Suppose $\sum_0^\infty a_n$ converges. Then $f(x) = \sum_0^\infty a_n x^n$ converges for $|x| < 1$ and $\lim_{x \to 1^-} f(x) = \sum_0^\infty a_n$.*

Proof. By general theorems on power series $f(x) = \sum_n^\infty a_n x^n$ converges for $|x| < 1$. Let $s_n = a_1 + \dots a_n$ and let $s = \lim_{n \to \infty} s_n = \sum_0^\infty a_n$. Then, by comparison to the geometric series $\sum s_n x^n$ converges for $|x| < 1$ and

$$f(x) = \sum_0^\infty a_n x^n = \sum_0^\infty s_n x^n - x \sum_0^\infty s_n x^n = (1 - x) \sum_0^\infty s_n x^n.$$

Then

$$f(x) - s = (1 - x) \sum_0^\infty (s_n - s) x^n,$$

since $\sum x^n = \frac{1}{1-x}$.

Now choose N so that if $n \geq N$ $|s_n - s| < \varepsilon/2$. Then

$$\begin{aligned} |f(x) - s| &= |(1 - x) \sum_0^\infty (s_n - s) x^n| \\ &\leq |(1 - x) \sum_0^N (s_n - s) x^n| + |(1 - x) \sum_{N+1}^\infty (s_n - s) x^n| \\ &\leq |(1 - x) \sum_0^N (s_n - s) x^n| + \varepsilon/2. \end{aligned}$$

Now we choose x close enough to 1 that

$$|(1 - x) \sum_0^N (s_n - s) x^n| \leq \varepsilon/2.$$

We can do this since $(1-x) \sum_0^N (s_n - s) x^n$ is a polynomial of degree N that vanishes at 1. We are done. $\square$

Let f be a continuous function on $[0, \infty)$ that doesn't grow too fast and is integrable on all intervals $[0, b]$ (for instance a polynomial). Then the integral

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

converges and defines a continuous function if $s > 0$. We don't need all of this. I am oversimplifying.

Theorem 2. *Suppose* $L = \int_0^\infty f(t)dt$ *exists. Then*

$$\lim_{s\to 0} F(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f(t)dt = L = \int_0^\infty f(t)dt$$

Proof. Let $G(s) = \int_0^s f(t)dt$. Then integration by parts gives (when $s > 0$)

$$\begin{aligned} F(s) &= e^{-st}G(t)|_0^\infty + s\int_0^\infty e^{-st}G(t)dt \\ &= s\int_0^\infty e^{-st}G(t)dt. \end{aligned}$$

Then

$$\begin{aligned} F(s) - L &= s\int_0^\infty e^{-st}G(t)dt - s\int_0^\infty e^{-st}Ldt \\ &= s\int_0^\infty e^{-st}(G(t) - L)dt \\ &= s\int_0^B e^{-st}(G(t) - L)dt + s\int_B^\infty e^{-st}(G(t) - L)dt. \end{aligned}$$

Choose B so that $|G(t) - L| < \varepsilon/2$ when $t \geq B$. And then choose s so that $|s\int_0^B e^{-st}(G(t) - L)dt| < \varepsilon/2$ when s is close to 0. We can do this since $s\int_0^B e^{-st}(G(t) - L)dt$ is continuous and is equal to 0 at $s = 0$. □

Remark 1. *This argument is general. We can use it discuss various methods of* summation. *Suppose* $a_n \to a$. *Then* $\frac{a_1+a_2+\cdots+a_n}{n} \to a$.

Proof. Let N be so large that $|a_n - a| < \varepsilon/2$ if $n \geq N$ and let $n = N + k$.

$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_n}{n} - a &= \frac{(a_1 - a) + (a_2 - a) + \cdots + (a_n - a)}{n} \\ &= \frac{(a_1 - a) + (a_2 - a) + \cdots + (a_N - a)}{N + k} + \frac{(a_{N+1} - a) + \cdots + (a_n - a)}{n} \\ &\leq \frac{(a_1 - a) + (a_2 - a) + \cdots + (a_N - a)}{N + k} + \frac{k}{n}\varepsilon/2 \\ &\leq \frac{(a_1 - a) + (a_2 - a) + \cdots + (a_N - a)}{N + k} + \varepsilon/2. \end{aligned}$$

With this fixed N, choose k and hence n so large that the first term is less than $\varepsilon/2$. □

Corollary 1. *Let* $s_n = a_1 + \cdots + a_n$. *Suppose* $s_n \to s$. *Then (Cesaro summation)* $\lim_{n\to\infty} \frac{s_1+\cdots+s_n}{n} \to s$.

Example 1. *For the example of* $\int_0^\infty \frac{\sin x}{x}dx$, *which we know converges, we compute the Laplace transform* ($s > 0$)

$$\int_0^\infty e^{-st}\frac{\sin t}{t}dt = \arctan(\frac{1}{s}),$$

and let $s \to 0+$.

References

[1] Niels Abel, *Untersuchungen uber die Reihe:*, pp. 311-339, Theorem IV, *Journal fur Math*, 1, (1826).