Abel's Theorem on Fourier Series

March 2, 2011

Abel's theorem allows us to conclude that if the Fourier coefficients $\hat{f}(n) = c_n$ are known and f is piecewise continuous then f is determined.

Definition 1. *Let* $0 < r < 1$ *and define*

$$A_r f(x) = \sum_{-\infty}^{+\infty} c_n r^{|n|} e^{inx}.$$

This series converges absolutely and uniformly in x to a continuous function of x for each $r < 1$.

Theorem 1. *If* f *is piecewise continuous*

$$\lim_{r \to 1^-} A_r f(x) = \frac{1}{2}[f(x^+) + f(x^-)].$$

If f *is continuous, given an* ϵ*, there is a* δ *so that*

$$|A_r f(x) - f(x)| < \epsilon$$

for all r *such that* $|r - 1| < \delta$*. We say* $A_r f$ *converges uniformly in* x *to* f*.*

Proof. Let $P_r(t) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} r^{|n|} e^{int}$. Let $z = re^{it}$. Then

$$\begin{aligned} P_r(x) &= \frac{1}{2\pi}\left[\frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}}\right] \\ &= \frac{1}{2\pi}\left[\frac{1-|z|^2}{|1-z|^2}\right] \\ &= \frac{1}{2\pi}\left[\frac{1-r^2}{1+r^2-2r\cos(t)}\right], \text{ and} \\ &= \frac{1}{2\pi}[1 + \sum_{n=1}^{\infty} 2r^n \cos(nt)] \end{aligned}$$

Integrating the last series term-by-term with respect to t we get

$$\int_0^{\pi} P_r(t)dt = \int_{-\pi}^{0} P_r(t)dt = \frac{1}{2}.$$

Now let $\delta > 0$ and suppose $\delta \leq t \leq \pi$. By calculus we find that the minimum of $1 + r^2 - 2r\cos(t)$ on this interval is $1 + r^2 - 2r\cos(\delta)$. Hence on $\delta \leq t \leq \pi$

$$0 < P_r(t) \leq \frac{1}{2\pi}\left[\frac{1-r^2}{1+r^2-2r\cos(\delta)}\right]. \tag{1}$$

Let us change variables and use periodicity, as in Dirichlet's theorem to write

$$A_r f(x_0) = \int_{-\pi}^{\pi} f(x_0 + t) P_r(t) dt.$$

Fix x_0 and choose δ so that $|f(x_0 + t) - f(x_0^-)| \leq \epsilon$ if $-\delta \leq t < 0$ and $|f(x_0 + t) - f(x_0^+)| \leq \epsilon$ if $0 < t \leq \delta$. Now that δ has been chosen, pick μ so that $0 \leq P_r(t) < \epsilon$ if $0 < 1 - r < \mu$ when $\delta \leq |t| \leq \pi$, which we can do by (1). Then

$$\begin{aligned} A_r f(x_0) - \frac{1}{2}[f(x^+) + f(x^-) &= \int_{-\pi}^{-\delta} [f(x_0 + t) - f(x_0^-)] P_r(t) dt + \int_{-\delta}^{0} [f(x_0 + t) - f(x_0^-)] P_r(t) dt \\ &+ \int_{0}^{\delta} [f(x_0 + t) - f(x_0^-)] P_r(t) dt + \int_{\delta}^{\pi} [f(x_0 + t) - f(x_0^-)] P_r(t) dt \\ &= I + II + III + IV. \end{aligned}$$

We'll first estimate III. The estimate on II is similar.

$$|III| \leq \epsilon \int_{0}^{\delta} P_r(t) dt \leq \epsilon \int_{0}^{\pi} P_r(t) dt = \frac{\epsilon}{2}.$$

Next we estimate I (IV is similar).

$$|I| \leq |\int_{-\pi}^{-\delta} [f(x_0 + t) - f(x_0^-)] P_r(t) dt| \leq 2M|\pi - \delta|\epsilon \leq 2\pi M\epsilon,$$

where $|f| \leq M$. So altogether we get

$$|A_r f(x_0) - \frac{1}{2}[f(x^+) + f(x^-)| \leq \epsilon + 4\pi M\epsilon,$$

when $0 < 1 - r < \mu$. This proves the first statement. The δ chosen depends on x_0 and hence μ depends on x_0. But if f is continuous on $[-\pi, \pi]$ it is uniformly continuous, so δ can be chosen independent of x_0 and then μ does not depend on x_0. $A_r f(x)$ is uniformly close to $f(x)$ if r is close enough to 1.

□