Math 335 Sample Problems

One notebook-sized page of notes (both sides may be used) will be allowed on the final exam. The final will be comprehensive.

1. Prove that
$$\int_0^1 (1-t^4)^{-1/2} dt = \frac{\Gamma(\frac{5}{4})\sqrt{\pi}}{\Gamma(\frac{3}{4})}.$$

2. Let f be a 2π-periodic function and let a be a fixed real number and let a new function g be defined by $g(x) = f(x-a)$. What is the relation between the Fourier coefficients $\widehat{f}(n)$ and $\widehat{g}(n)$?

3. Find the Fourier series of
$$\frac{1-r^2}{1-2r\cos x + r^2}$$
where $0 \le r < 1$. (You don't need to integrate.)

4. Let f be a 2π-periodic, piecewise smooth function. Let $\widehat{f}(n)$ be the complex Fourier coefficients of f. Show that there is a constant M (which will depend on f) such that $|\widehat{f}(n)| < M/|n|$ for all $n \neq 0$. Do **not** assume f is continuous.

5. Suppose f_k is a sequence of Riemann integrable functions on $[0, 2\pi]$ such that $\lim_{k\to\infty} \int_0^{2\pi} |f_k - f| = 0$. Prove that the Fourier coefficients satisfy $\lim_{k\to\infty} \widehat{f_k}(n) = \widehat{f}(n)$ for each n.

6. Suppose f and g are 2π-periodic and Riemann integrable on compact subsets of $\mathbf{R}$. Suppose also that $f(x) = g(x)$ in a neighborhood of a point x_0. Suppose that the Fourier series for one of the functions converges at x_0. Prove that the other series converges and
$$\sum_{-\infty}^{\infty} \widehat{f}(n)e^{inx_0} = \sum_{-\infty}^{\infty} \widehat{g}(n)e^{inx_0}.$$
Hint: Look at $f - g$.

7. Prove that
$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin(nx)}{x} dx \;=\; \frac{\pi}{2}.$$

8. Define a function $\log_p(x)$ inductively by the formulas $\log_0(x) = x, \log_{p+1}(x) = \log(\log_p(x))$. Prove by induction that the series
$$\sum_{n=m}^{\infty} \frac{1}{\log_0(n)\log_1(n)\log_2(n)\ldots\log_p(n)}$$

(where m is large enough for the denominators to be defined as real numbers) diverges for every p.

9. Suppose that $a_n > 0$, that a_n is decreasing, and that $\sum_1^\infty a_n$ converges. Is it true that $\lim_{n\to\infty} na_n = 0$? If true prove it, if false give a counterexample.

10. Show that the series $\sum_1^\infty \frac{\sin nx}{\sqrt{n}}$ converges for all x and uniformly on any interval of the form $[\delta, 2\pi - \delta]$, where $\delta > 0$ is small. Show that the series is not the Fourier series of a Riemann integrable function.

11. Find the solution of $u_t = 3u_{xx},\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = \cos x \sin 5x$. (This is easier than it looks.)

12. (a) Let $\sum_0^\infty a_n x^n$ be a series with radius of convergence R. Substitute $re^{i\theta}$ for x and get a new series involving $e^{in\theta}$. If $0 < r < R$ prove that this is a Fourier series (the variable is θ).
    (b) Prove that $\sum_0^\infty r^{2n}|a_n|^2$ converges for $0 \le r < R$.

13. Compute

$$\lim_{n\to\infty} \int_a^b \sin^2(nx)dx.$$

14. Folland, §8.6: problem 10.

15. Let f and g be continuous 2π-periodic functions. Define the *convolution* of f and g to be the function. $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)dt$.
    (a) Prove that $f * g$ is 2π-periodic.
    (b) Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$, so the Fourier series of $f * g$ is $\sum_{-\infty}^{\infty} c_n d_n e^{inx}$, where $c_n = \widehat{f}(n),\ d_n = \widehat{g}(n)$.

16. (a) Find the cosine series of f where $f(x) = 0, 0 < x < \pi/2;\ f(x) = 1, \pi/2 < x < \pi$.
    (b) Prove that the series converges for all x.
    (c) For which x does the series converge absolutely?

17. (a) Let $r = \sqrt{x^2 + y^2}$. Prove that $\frac{y}{r^2}$ is harmonic when $y > 0$.

(b) Suppose $\phi(t)$ is continuous on $[a, b]$. Let

$$u(x, y) = \int_a^b \frac{\phi(t) y dt}{(x-t)^2 + y^2}.$$

Prove that u is harmonic when $y > 0$.

18. let f be 2π-periodic, continuous, and piecewise smooth. Let m be any positive integer and define the function f_m by the formula $f_m(x) = f(mx)$. Prove that $\widehat{f_m}(n) = \widehat{f}\left(\frac{n}{m}\right)$ if m divides n and is 0 otherwise.

19. Determine a, b, c so that $f_0(x) = 1, f_1(x) = x + a, f_2(x) = x^2 + bx + c$ is an orthogonal set using the inner product $< f, g >= \int_0^2 fg$ on $[0, 2]$.

20. (Extra Credit) This is a "counterexample" to the Cantor-Lebesgue theorem. Let $n_j = \frac{j(j+1)}{2}$ so that $n_j - n_{j-1} = j$, and consider the series, $\sum_{j=1}^{\infty} \sin(2^{n_j} x)$. let $E = \{2\pi c\}$, where c is written in binary notation and is of the form $\sum_{j=1}^{\infty} e_j 2^{-n_j},\ e_j \in \{0, 1\}$. Prove that $\sum \sin(2^{n_j} x)$ converges uniformly and absolutely on E, but the coefficients don't go to 0. E is an uncountable set.

21. (Extra, extra credit) Let (x) be the function with period 1 that equals x on $(-1/2, 1/2)$ and equals 0 at $\pm 1/2$. Define a function f as follows

$$f(x) = \sum_{1}^{\infty} \frac{(nx)}{n^2}.$$

This is an example of Riemann (not published until after his death).

(a) Prove that the series defining (1) converges uniformly on $\mathbb{R}$.

(b) Prove that f is continuous except at points of the form $\frac{2s+1}{2n}$. Prove that if $2s+1$ and n are relatively prime there is a jump discontinuity of size $\frac{-\pi^2}{8n^2}$ at $\frac{2s+1}{2n}$.

(c) Prove that f is Riemann integrable on each compact subinterval of $\mathbb{R}$.

22. There may be problems from the text, statements of theorems from the text, problems from previous review sets, or examples from class on the exam.