Double Series

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This note is a brief discussion of double series.

Definition 1. *Suppose* $\{a_{m,n}\}$ *is a double sequence.* $\lim_{m,n\to\infty} a_{m,n} = a$ *means that for every* $\epsilon > 0$ *there is an integer* K *so that if* $m \geq K$ *and* $n \geq K$ *then* $|a_{m,n} - a| \leq \epsilon$.

Theorem 1. *If* $\lim_{m,n\to\infty} a_{m,n} = a$ *and* $\lim_{n\to\infty} a_{m,n}$ *exists for every* m *then* $\lim_{m\to\infty}(\lim_{n\to\infty} a_{m,n}) = a$.

Proof. Let $b_m = \lim_{n\to\infty} a_{m,n}$. For large enough n, m $|a_{m,n} - a| < \epsilon$. Hence $|\lim_{n\to\infty} a_{m,n} - a| = |b_m - a| \leq \epsilon$. Hence $|\lim_{n\to\infty} b_m - a| \leq \epsilon$. Since ϵ is arbitrary, $\lim_{m\to\infty}(\lim_{n\to\infty} a_{m,n}) = a$. □

Definition 2. *Let* $\sum_{m,n} a_{m,n}$ *be an infinite series. Let* $\sum_{j=1,k=1}^{j=m,k=n} a_{j,k} = s_{m,n}$. $\sum_{m,n} a_{m,n}$ ***converges*** *if* $\lim_{m,n\to\infty} s_{m,n}$ *exists.*

Corollary 1. *If* $\sum_{m,n} a_{m,n}$ *converges and* $\sum_{n=1}^{\infty} a_{m,n}$ *exists for all* m, *then*

$$\sum_{m,n} a_{m,n} = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{m,n}).$$

Definition 3. $\sum_{m,n} a_{m,n}$ ***converges absolutely*** *if* $\sum_{m,n} |a_{m,n}|$ *converges.*

Theorem 2. *If* $\sum_{m,n} a_{m,n}$ *converges absolutely then* $\sum_{m,n} a_{m,n}$ *converges and the sum can be computed by any arrangement of the terms.*

This theorem is proved in the same way as the theorem for singly indexed series.

Theorem 3. *If* $\sum_m^{\infty}(\sum_n^{\infty} |a_{m,n}|)$ *converges then* $\sum a_{m,n}$ *converges absolutely and* $\sum a_{m,n} = \sum_m^{\infty}(\sum_n^{\infty} a_{m,n})$. *We can compute the sum* $\sum a_{m,n}$ *in any order.*

Proof. $\sum_{j,k}^{m,n} |a_{j,k}| \leq \sum_j^m \sum_k^{\infty} |a_{j,k}| \leq \sum_j^{\infty}(\sum_k^{\infty} |a_{j,k}|)$, so $\sum_{m,n} a_{m,n}$ converges absolutely, hence converges. Now we can use Theorem 1 to complete the proof. □

Theorem 4. *Suppose* $\sum_n a_n$ *and* $\sum_n b_n$ *converge absolutely. Let* $A = \sum a_n$, $B = \sum b_n$, $c_n = \sum_{j+k=n} a_j b_k$. *Then* $AB = \sum_n c_n$.

Proof. Let $d_{j,k} = a_j b_k$. Then $\sum_j(\sum_k |d_{j,k}|)$ converges, hence $AB = \sum_j(\sum_k d_{j,k}) = \sum c_n$ since we can compute the sum in any order. □