Math 335 Sample Problems

One notebook sized page of notes (one side) will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5, 4.6, and chapters 6 and 7. The midterm is on Monday, February 11. We have covered a lot of material. This is an extensive list of problems but I will ask only easy versions of these problems. I will not ask questions from sections 4.5, 7.5, and 7.6.

1. Let \( f(x) \) satisfy \( 0 \leq f(x) \leq f(y) \) if \( x \geq y \). Suppose \( \int_1^\infty f(x) \, dx \) converges. Prove \( \lim_{x \to +\infty} xf(x) = 0 \).

2. Assume \( a_n \geq 0 \) for all \( n \geq 1 \). Prove that if \( \sum_{1}^{\infty} a_n \) converges then \( \sum_{1}^{\infty} \sqrt{a_n a_{n+1}} \) converges. Give an example of a sequence \( a_n \geq 0 \) such that \( \sum_{1}^{\infty} \sqrt{a_n a_{n+1}} \) converges and \( \sum_{1}^{\infty} a_n \) diverges.

3. Prove that if \( \sum_{1}^{\infty} a_n \) converges then \( \sum_{1}^{\infty} \frac{\sqrt{a_n}}{n} \) converges. (Assume \( a_n \geq 0 \).)

4. Let \( x_n \) be a convergent sequence and let \( c = \lim_{n \to \infty} x_n \). Let \( p \) be a fixed positive integer and let \( a_n = x_n - x_{n+p} \). Prove that \( \sum a_n \) converges and

\[
\sum_{1}^{\infty} a_n = x_1 + x_2 + \ldots + x_p - pc.
\]
5. Suppose \(a_n > 0, \ b_n > 0\) for all \(n > 1\). Suppose that \(\sum_{1}^{\infty} b_n\) converges and that \(\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}\) for \(n \geq N\). Prove that \(\sum_{1}^{\infty} a_n\) converges.

6. Let \(S\) be the set of all positive integers whose decimal representation does not contain 2. Prove that \(\sum_{n \in S} \frac{1}{n}\) converges.

7. Prove that \(\int_{0}^{\infty} \cos x^2 \, dx\) converges, but not absolutely.

8. Let \(a = \lim_{n \to \infty} a_n\). Prove that \(\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = a\).

9. Decide if the following integrals converge conditionally, converge absolutely, or diverge.
   (a) \(\int_{-\infty}^{+\infty} x^2 e^{-|x|} \, dx\)
   (b) \(\int_{0}^{\pi} \frac{dx}{(\cos x)^{3/2}}\)
   (c) \(\int_{1}^{\infty} \frac{\sin(1/x)}{x} \, dx\)

10. Let \(f\) and \(g\) be integrable on \([a, b]\) for every \(b > a\).
   (a) Prove that \((\int_{a}^{b} |fg|)^2 \leq \int_{a}^{b} f^2 \int_{a}^{b} g^2\).
   (b) Prove that if \(\int_{a}^{\infty} f^2\) and \(\int_{a}^{\infty} g^2\) converge then \(\int_{a}^{\infty} fg\) converges absolutely.
11. (a) Suppose \( \sum_1^{\infty} a_n \) converges. Fix \( p \in \mathbb{Z}^+ \). Prove that \( \lim_{n \to \infty} (a_n + a_{n+1} + \ldots + a_{n+p}) = 0. \)

(b) Suppose \( \lim_{n \to \infty} (a_n + a_{n+1} + \ldots + a_{n+p}) = 0 \) for every \( p \). Does \( \sum_1^{\infty} a_n \) converge?

12. Let \( a_n > 0 \) and suppose \( a_n \geq a_{n+1} \). Prove that \( \sum_1^{\infty} a_n \) converges if and only if \( \sum_0^{\infty} a_{3n} \) converges.

13. Suppose \( f_n \) is a sequence of continuous functions that converges uniformly on a set \( W \). Let \( p_n \) be a sequence of points in \( W \) that converges to a point \( p \in W \). Prove that \( \lim_{n \to \infty} f_n(p_n) = f(p) \).

14. Prove that \( \sum_{n=0}^{\infty} \frac{x}{(1 + |x|)^n} \) converges for all \( x \), but the convergence is not uniform.

15. Assume \( p \geq 1, \ q \geq 1 \). Prove that
\[
\int_0^1 \frac{t^{p-1}}{1 + t^q} \, dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} - \ldots.
\]
Give careful justification of any manipulations.

16. Suppose \( a_n > b_n > 0, \ a_n > a_{n+1} \) and \( \lim_{n \to \infty} a_n = 0 \). Does \( \sum_1^{\infty} (-1)^n b_n \) converge? Give a proof or a counterexample.

17. Prove that \( \sum_{n=1}^{\infty} \frac{\cos n \pi}{n} \) converges uniformly for \( x \in [a, b], 0 < a < b < 2\pi \), but does not converge absolutely for any \( x \).

18. Prove that \( \sum_{1}^{\infty} (-1)^n \frac{\sin n \pi}{n} \) converges uniformly on \( \{|x| < 1\} \) to a continuous function.
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19. Let $f_n$ be a sequence of functions defined on the open interval $(a, b)$. Suppose $\lim_{x \to a^+} f_n(x) = a_n$ for all $n$. Suppose $\sum_{1}^{\infty} f_n$ converges uniformly on $(a, b)$ to a function $f$. Prove that $\sum_{1}^{\infty} a_n$ converges and $\lim_{x \to a^+} f(x) = \sum_{1}^{\infty} a_n$. Do not assume $f_n$ is continuous on $(a, b)$.

20. Suppose the series $\sum_{1}^{\infty} a_n$ converges. Prove that $\sum_{1}^{\infty} \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_{1}^{\infty} \frac{a_n}{n^x}$. Prove that $\lim_{x \to 0^+} f(x) = \sum_{1}^{\infty} a_n$.

21. We have covered the following:

(a) Improper single and multiple integrals
(b) Convergence and divergence of a series
(c) Comparison test
(d) Integral test
(e) Cauchy condensation test
(f) Root test and ratio test
(g) Absolute and conditional convergence of a series
(h) Dirichlet’s test
(i) Abel’s test and theorem
(j) Uniform convergence of a sequence or series of functions
(k) Weierstrass M-test
(l) Continuity of a uniform limit of continuous functions
(m) Integration and differentiation of a sequence or series
(n) Power series
(o) Radius of convergence of a power series
(p) Integration and differentiation of a power series

22. There may be homework problems or example problems from the text on the midterm.