Math 335 Sample Problems

One notebook sized page of notes (*one side*)will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5, 4.6, and chapters 6 and 7. The midterm is on Monday, February 11. We have covered a lot of material. This is an extensive list of problems but I will ask only easy versions of these problems. I will not ask questions from sections 4.5, 7.5, and 7.6.

1. Let $f(x)$ satisfy $0 \leq f(x) \leq f(y)$ if $x \geq y$. Suppose $\displaystyle\int_1^\infty f(x)dx$ converges. Prove $\displaystyle\lim_{x\to+\infty} xf(x) = 0$.

2. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\displaystyle\sum_1^\infty a_n$ converges then $\displaystyle\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\displaystyle\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\displaystyle\sum_1^\infty a_n$ diverges.

3. Prove that if $\displaystyle\sum_1^\infty a_n$ converges then $\displaystyle\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

4. Let x_n be a convergent sequence and let $c = \displaystyle\lim_{n\to\infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and
$$\sum_1^\infty a_n = x_1 + x_2 + \dots x_p - pc.$$

5. Suppose $a_n > 0,\ b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

6. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n \in S} \frac{1}{n}$ converges.

7. Prove that $\int_0^\infty \cos x^2 dx$ converges, but not absolutely.

8. Let $a = \lim_{n\to\infty} a_n$. Prove that $\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = a$.

9. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

   (a)
   $$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

   (b)
   $$\int_0^\pi \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

   (c)
   $$\int_1^\infty \frac{\sin(1/x)}{x} dx$$

10. Let f and g be integrable on $[a, b]$ for every $b > a$.

    (a) Prove that
    $$\left(\int_a^b |fg|\right)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

    (b) Prove that if $\int_a^\infty f^2$ and $\int_a^\infty g^2$ converge then $\int_a^\infty fg$ converges absolutely.

11. (a) Suppose $\sum_1^\infty a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n\to\infty}(a_n + a_{n+1} + \dots a_{n+p}) = 0$.
    (b) Suppose $\lim_{n\to\infty}(a_n + a_{n+1} + \dots a_{n+p}) = 0$ for every p. Does $\sum_1^\infty a_n$ converge?

12. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\displaystyle\sum_1^\infty a_n$ converges if and only if $\displaystyle\sum_0^\infty a_{3^n}$ converges.

13. Suppose f_n is a sequence of continuous functions that converges uniformly on a set W. Let p_n be a sequence of points in W that converges to a point $p \in W$. Prove that $\lim_{n\to\infty} f_n(p_n) = f(p)$.

14. Prove that $\displaystyle\sum_{n=0}^\infty \frac{x}{(1+|x|)^n}$ converges for all x, but the convergence is not uniform.

15. Assume $p \geq 1,\ q \geq 1$. Prove that
$$\int_0^1 \frac{t^{p-1}}{1+t^q}dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} \cdots .$$
Give careful justification of any manipulations.

16. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\displaystyle\lim_{n\to\infty} a_n = 0$. Does $\displaystyle\sum_1^\infty (-1)^n b_n$ converge? Give a proof or a counterexample.

17. Prove that $\displaystyle\sum_{n=1}^\infty \frac{\cos nx}{n}$ converges uniformly for $x \in [a,b], 0 < a < b < 2\pi$, but does not converge absolutely for any x.

18. Prove that $\displaystyle\sum_1^\infty (-1)^n \frac{\sin nx}{n}$ converges uniformly on $\{|x| < 1\}$ to a continuous function.

19. Let f_n be a sequence of functions defined on the open interval (a, b). Suppose $\lim_{x \to a^+} f_n(x) = a_n$ for all n. Suppose $\sum_1^\infty f_n$ converges uniformly on (a, b) to a function f. Prove that $\sum_1^\infty a_n$ converges and $\lim_{x \to a^+} f(x) = \sum_1^\infty a_n$. Do not assume f_n is continuous on (a, b).

20. Suppose the series $\sum_1^\infty a_n$ converges. Prove that $\sum_1^\infty \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_1^\infty \frac{a_n}{n^x}$. Prove that $\lim_{x \to 0^+} f(x) = \sum_1^\infty a_n$.

21. We have covered the following:
    (a) Improper single and multiple integrals
    (b) Convergence and divergence of a series
    (c) Comparison test
    (d) Integral test
    (e) Cauchy condensation test
    (f) Root test and ratio test
    (g) Absolute and conditional convergence of a series
    (h) Dirichlet's test
    (i) Abel's test and theorem
    (j) Uniform convergence of a sequence or series of functions
    (k) Weierstrass M-test
    (l) Continuity of a uniform limit of continuous functions
    (m) Integration and differentiation of a sequence or series
    (n) Power series
    (o) Radius of convergence of a power series
    (p) Integration and differentiation of a power series

22. There may be homework problems or example problems from the text on the midterm.