Dirichlet's Test

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Theorem 1. *Let* $a_n \geq 0$ *be a decreasing sequence,* $a_n \geq a_{n+1}$ *and* $a_n \to 0$*. Suppose there is a number* M *so that* $|\sum_1^n b_n| \leq M$ *for all* n*. Then* $\sum_1^\infty a_n b_n$ *converges.*

Proof. Define $B_n = b_1 + b_2 + \cdots + b_n, B_0 = 0$. Then $b_n = B_n - B_{n-1}$ and

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 + \dots a_nb_n &= a_1(B_1 - B_0) + a_2(B_2 - B_1) + a_3(B_3 - B_2) + \dots a_{n-1}(B_{n-1} - B_{n-2}) + a_n(B_n - B_{n-1}) \\ &= B_1(a_1 - a_2) + B_2(a_2 - a_3) + \dots B_{n-1}(a_{n-1} - a_n) + B_n a_n. \end{aligned}$$

Let

$$s_n = \sum_{k=1}^{n} a_k b_k \text{ and } S_{n-1} = \sum_{k=1}^{n-1} B_k(a_k - a_{k+1}).$$

Then the preceding equation can be written as $s_n = S_{n-1} + B_n a_n$. Since $|B_n| \leq M$ and $a_n \to 0$, $B_n a_n \to 0$. Now I claim that the series $\sum_{k=1}^n B_k(a_k - a_{k+1})$ converges absolutely. We estimate $|B_k(a_k - a_{k+1})| \leq M|a_k - a_{k+1}| = M(a_k - a_{k+1})$ since $a_k - a_{k+1} \geq 0$. The partial sums of the series $\sum_1^n (a_k - a_{k+1})$ are $a_1 - a_{n+1}$. Since $a_n \to 0$ the series converges to a_1 and by comparison the series $\sum_{k=1}^n B_k(a_k - a_{k+1})$ converges absolutely. We did not prove that $\sum_1^\infty a_n b_n$ converges absolutely. We related the partial sums to the series $\sum_{k=1}^n B_k(a_k - a_{k+1})$ which converges absolutely.

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