Math 335 Sample Problems

One notebook sized page of notes (one side) will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5, 4.6, 4.7, 5.6, 5.7, 5.8, 6.1, and 6.2. The midterm is on Monday, January 30.

1. Let $f(x)$ satisfy $0 \leq f(x) \leq f(y)$ if $x \geq y$. Suppose $\int_1^{\infty} f(x) \, dx$ converges. Prove $\lim_{x \to +\infty} xf(x) = 0$.

2. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges and $\sum_{n=1}^{\infty} a_n$ diverges.

3. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n \, n}$ converges. (Assume $a_n \geq 0$.)

4. Let $x_n$ be a convergent sequence and let $c = \lim_{n \to \infty} x_n$. Let $p$ be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and

$$
\sum_{n=1}^{\infty} a_n = x_1 + x_2 + \ldots + x_p - pc.
$$
5. Suppose $a_n > 0$, $b_n > 0$ for all $n > 1$. Suppose that $\sum_{1}^{\infty} b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_{1}^{\infty} a_n$ converges.

6. Let $S$ be the set of all positive integers whose decimal representation does not contain 2. Prove that $\sum_{n \in S} \frac{1}{n}$ converges.

7. Prove that $\int_{0}^{\infty} \cos x^2 dx$ converges, but not absolutely.

8. Let $a = \lim_{n \to \infty} a_n$. Prove that $\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = a$.

9. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

   (a) $\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$

   (b) $\int_{0}^{\pi} \frac{dx}{(\cos x)^\frac{3}{2}}$

   (c) $\int_{1}^{\infty} \frac{\sin(1/x)}{x} dx$

10. Let $f$ and $g$ be integrable on $[a, b]$ for every $b > a$.

    (a) Prove that $\left( \int_{a}^{b} |fg| \right)^2 \leq \int_{a}^{b} f^2 \int_{a}^{b} g^2$.

    (b) Prove that if $\int_{a}^{\infty} f^2$ and $\int_{a}^{\infty} g^2$ converge then $\int_{a}^{\infty} fg$ converges absolutely.
11. (a) Suppose $\sum_{1}^{\infty} a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n \to \infty} (a_n + a_{n+1} + \ldots a_{n+p}) = 0$. 
(b) Suppose $\lim_{n \to \infty} (a_n + a_{n+1} + \ldots a_{n+p}) = 0$ for every $p$. Does $\sum_{1}^{\infty} a_n$ converge?

12. Let $C$ be the curve of intersection of $y+z=0$ and $x^2+y^2=a^2$ oriented in the counterclockwise direction when viewed from a point high on the $z$-axis. Use Stokes’ theorem to compute the value of $\int_C (xz+1)dx + (yz+2x)dy$.

13. (a) Prove that $\int_C \frac{-ydx + xdy}{x^2 + y^2}$ is not independent of path on $\mathbb{R}^2 - 0$.
(b) Prove that $\int_C \frac{xdx + ydy}{x^2 + y^2}$ is independent of path on $\mathbb{R}^2 - 0$. Find a function $f(x,y)$ on $\mathbb{R}^2 - 0$ so that $\nabla f = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$.

14. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\sum_{1}^{\infty} a_n$ converges if and only if $\sum_{0}^{\infty} a_{3n}$ converges.

15. Suppose that $a_n > 0$ is a sequence of positive numbers and suppose that the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ exists. Then prove that $\lim_{n \to \infty} \sqrt[n]{a_n}$ exists and 
$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \sqrt[n]{a_n}$$

16. You will need to know the definitions of the following terms and statements of the following theorems.

(a) Convergence and divergence of a series
(b) Comparison test
(c) Integral test
(d) Cauchy condensation test
(e) Root test and ratio test
(f) Stokes’ theorem
Sample Problems

(g) Potentials and independence of path
(h) Poincare’s lemma
(i) Improper single and multiple integrals
(j) Integrals dependent on a parameter

17. There may be homework problems or example problems from the text on the midterm.