Riemann-Lebesgue Lemma

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The Riemann-Lebesgue lemma is quite general, but since we only know Riemann integration, I’ll state it in that form.

Theorem 1. Let \( f \) be Riemann integrable on \([a, b]\). Then

\[
\lim_{\lambda \to \pm\infty} \int_a^b f(t) \cos(\lambda t) \, dt = 0 \tag{1}
\]

\[
\lim_{\lambda \to \pm\infty} \int_a^b f(t) \sin(\lambda t) \, dt = 0 \tag{2}
\]

\[
\lim_{\lambda \to \pm\infty} \int_a^b f(t) e^{i\lambda t} \, dt = 0 \tag{3}
\]

Proof. I will prove only the first statement. Since \( f \) is integrable, given \( \epsilon > 0 \), there is a partition \( \{a = x_0, x_1, \ldots, x_n = b\} \), so that \( \epsilon/2 > \int_a^b f - \sum_1^n m_i \Delta x_i \geq 0 \), where \( m_i \) is the minimum of \( f \) on \([x_{i-1}, x_i]\).

But the sum can be written as \( \sum_1^n m_i \Delta x_i = \int_a^b g \), where \( g = \sum m_i \chi_{[x_{i-1}, x_i]} \), and the inequality takes the form

\[
\epsilon/2 > \int_a^b (f - g) \geq 0.
\]

Now we use the fact that \( f - g \geq 0 \) to get

\[
\left| \int_a^b f(t) \cos(\lambda t) \, dt \right| \leq \left| \int_a^b (f(t) - g(t)) \cos(\lambda t) \, dt \right| + \left| \int_a^b g(t) \cos(\lambda t) \, dt \right| \tag{4}
\]

\[
\leq \int_a^b (f - g) + \left| (1/\lambda) \sum m_i (\sin(\lambda x_i) - \sin(\lambda x_{i-1})) \right|. \tag{5}
\]

The function \( g \) has been fixed. Take \( \lambda \) large enough that

\[
\left| (1/\lambda) \sum m_i (\sin(\lambda x_i) - \sin(\lambda x_{i-1})) \right| < \epsilon/2,
\]

and we are done.

This proof works nearly verbatim for Lebesgue integration and non-compact intervals.