Suppose $w$ is a solution of the following initial value problem for a linear differential equation of $n^{th}$ order with constant coefficients:

$$Lw = w^{(n)}(x) + a_{n-1}w^{(n-1)}(x) + \cdots + a_1w'(x) + a_0w(x) = 0$$  \hfill (1)

$$w(0) = 0, \hfill (2)$$

$$w'(0) = 0, \hfill (3)$$

$$\ldots, \hfill (4)$$

$$w^{(n-2)}(0) = 0, \hfill (5)$$

$$w^{(n-1)}(0) = 1. \hfill (6)$$

Let $k(x, y) = w(x - y)$, let $g$ be continuous, and set

$$h(x) = \int_0^x k(x, y)g(y)dy = \int_0^x w(x - y)g(y)dy.$$
Let’s compute:

\[ h(0) = 0, \tag{7} \]
\[ h'(x) = w(0)g(x) + \int_0^x w'(x - y)g(y)dy, \tag{8} \]
\[ h'(x) = \int_0^x w'(x - y)g(y)dy, \tag{9} \]
\[ h'(0) = 0, \tag{10} \]
\[ h''(x) = w'(0)g(x) + \int_0^x w''(x - y)g(y)dy, \tag{11} \]
\[ h''(x) = \int_0^x w''(x - y)g(y)dy, \tag{12} \]
\[ h''(0) = 0, \tag{13} \]
\[ \ldots, \tag{14} \]
\[ h^{(n-1)}(x) = \int_0^x w^{(n-1)}(x - y)g(y)dy, \tag{15} \]
\[ h^{(n-1)}(0) = 0, \tag{16} \]
\[ h^{(n)}(x) = w^{(n-1)}(0)g(x) + \int_0^x w^{(n)}(x - y)g(y)dy, \tag{17} \]
\[ h^{(n)}(x) = g(x) + \int_0^x w^{(n)}(x - y)g(y)dy, \tag{18} \]
\[ Lh(x) = g(x) + \int_0^x Lw(x - y)g(y)dy, \tag{19} \]
\[ Lh(x) = g(x). \tag{20} \]

The last line is true because \( Lw = 0 \). The other lines are true because it is legal to differentiate under the integral sign and because of the chain rule, the fundamental theorem of calculus, and equations 1-6.

We have found a solution, represented by an integral of

\[ Lh(x) = g(x), \tag{21} \]
\[ h(0) = 0, \tag{22} \]
\[ h'(0) = 0, \tag{23} \]
\[ \ldots, \tag{24} \]
\[ h^{(n-1)}(0) = 0. \tag{25} \]

It is

\[ h(x) = \int_0^x w(x - y)g(y)dy. \]