The words *closed* and *exact* have many meanings. I will make up some terms that are private for this class.

**Definition 1.** A vector field $F$ is **curl-closed** if $\text{curl}(F) = 0$. $F$ is **gradient-exact** if $F = \text{grad}(f)$. $F$ is **div-closed** if $\text{div}(F) = 0$. $F$ is **curl-exact** if $F = \text{curl}(G)$.

**Theorem 1.** On a box with faces parallel to the axis planes,

$$\text{curl-closed} \iff \text{gradient-exact},$$

$$\text{div-closed} \iff \text{curl-exact}.$$  \hspace{1cm} (1)

**Proof.** I will only prove statement (2).

First, if $F = \text{curl}(G)$ where $G = (U,V,W)$. Then

$$\text{div}(F) = (W_y - V_z)x - (W_z - U_z)y + (V_x - U_y)z \hspace{1cm} (3)$$

$$= 0. \hspace{1cm} (4)$$

Next let $F = (P,Q,R)$ and suppose $P_x + Q_y + R_z = 0$. Suppose there is vector field $G = (U,V,W)$ so that $F = \text{curl}(G)$. If we add $\text{grad}(f)$ to $G$ then since $\text{curl}(\text{grad}(f)) = 0$, it is still true that $F = \text{curl}(G)$. Now we can always choose $f$ so that $f_z = -W$, in which case the $z$-component of $G + \text{grad}(f)$ is 0. In other words we can assume that $W = 0$. Now our requirements become

$$P = -V_z, \hspace{1cm} (5)$$

$$Q = U_z, \hspace{1cm} (6)$$

$$R = V_x - U_y. \hspace{1cm} (7)$$

We solve the first two equations by taking any $z$-antiderivative of $-P$ for $V$ and any $z$-antiderivative of $Q$ for $U$. In symbolic form

$$V(x,y,z) = -\int_{z_0}^{z} P(x,y,t)dt + \phi(x,y), \hspace{1cm} (8)$$

$$U(x,y,z) = \int_{z_0}^{z} Q(x,y,t)dt + \psi(x,y). \hspace{1cm} (9)$$

Where we have let $\phi(x,y) = V(x,y,z_0)$ and $\psi(x,y) = U(x,y,z_0)$. We can do this on a box. Now we need to solve (7). But (7) is

$$R(x,y,z) = -\int_{z_0}^{z} (Q_y + P_x)dt + \phi_x - \psi_y$$

$$= R(x,y,z) - R(x,y,z_0) + \phi_x - \psi_y. \hspace{1cm} (10)$$

We can solve this equation by letting $\psi = 0$ and choosing any solution of $\phi_x(x,y) = R(x,y,z_0)$. \hfill $\Box$
Example:
Let $\mathbf{F} = (3x^2y, -xy^2, -4xyz)$. Then check that $\text{div}(\mathbf{F}) = 0$. Equations 5, 6, 7 become

\begin{align*}
3x^2y &= -V_z, \quad (12) \\
-xy^2 &= U_z, \quad (13) \\
-4xyz &= V_x - U_y. \quad (14)
\end{align*}

Following the proof of the theorem we find that

\begin{align*}
U &= -zxy^2 + \phi(x, y), \quad (15) \\
V &= -3x^2yz + \psi(x, y). \quad (16)
\end{align*}

Let $\psi = 0$ and then find that equation 14 is

\begin{equation*}
-\phi_y = 0,
\end{equation*}

so we also choose $\phi = 0$. The solution is then

\begin{align*}
U &= -zxy^2, \quad (17) \\
V &= -3x^2yz, \quad (18) \\
W &= 0. \quad (19)
\end{align*}