Bump Functions

January 17, 2012

This note describes how to make a $C^\infty$ bump function with compact support. The exposition is taken from Jack Lee’s book, *Introduction to Smooth Manifolds*. We are already familiar with the function $f(x) = e^{-1/x^2}$, if $x \neq 0$; 0, if $x = 0$. See Folland exercise 9, §2.1. Now define a new $C^\infty$ function $h(x)$ by

$$h(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

The same argument used in exercise 9 can be used to prove that $h$ is $C^\infty$. We define a $C^\infty$ function $g$ by

$$g(x) = \frac{h(2 - x)}{h(2 - x) + h(x - 1)}.$$ 

Then

$$g(x) = 1, \text{ if } x < 1,$$

$$g(x) = 0, \text{ if } x > 2,$$

$$0 \leq g(x) \leq 1, \text{ if } 1 \leq x \leq 2.$$ 

Finally define

$$b(x) = g(|x|).$$

Then $b(x) = 0$ if $|x| > 2$, $b(x) = 1$ if $|x| < 1$, and $0 \leq b(x) \leq 1$ if $1 \leq |x| \leq 2$. Also $b$ is $C^\infty$, since it is clearly $C^\infty$ if $|x| < 1$ or $|x| > 2$; and since $|x|$ is $C^\infty$ when $x \neq 0$, $b$ is the composition of $C^\infty$ functions for $1 \leq |x| \leq 2$.

If we use a linear change of coordinates we can create a $C^\infty$ function

$$b_{a,b}(x) = b(-2 + 4 \frac{x-a}{b-a}),$$

such that $b_{a,b}(x) > 0$ in $(a,b)$ and $b_{a,b} = 0$ if $x \notin [a,b]$. 