# Spherical Coordinates

Note Title 1/15/2009

Spherical coordinates in $n$-space are defined by $x_1 = r\cos\phi_1$, $x_2 = r\sin\phi_1\cos\phi_2$, $x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3, \dots$

$x_{n-1} = r\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-2}\cos\theta,$

$x_n = r\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-2}\sin\theta.$

$0 \le \phi_j \le \pi$, $0 \le \theta \le 2\pi$.

Let's compute the Jacobian. I'll use the abbreviations $s_j = \sin\phi_j$, $c_j = \cos\phi_j$, $s = \sin\theta$, $c = \cos\theta$

Then $x_1 = rc_1$, $x_2 = rs_1c_2$, $x_3 = rs_1s_2c_3, \dots$

$x_{n-1} = rs_1s_2\cdots s_{n-2}c$, $x_n = rs_1s_2\cdots s_{n-2}\cdot s.$

Let

$$J_n(r) = \begin{bmatrix} c_1 & -rs_1 & 0 & 0 & 0 & \cdots & 0 \\ s_1c_2 & rc_1c_2 & -rs_1s_2 & 0 & \cdots & & 0 \\ s_1s_2c_3 & rc_1s_2c_3 & rs_1c_2c_3 & -rs_1s_2s_3 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ s_1s_2\cdots c & rc_1s_2\cdots c & & & & & -rs_1s_2\cdots s_{n-2}s \\ s_1s_2\cdots s & rc_1s_2\cdots s & & & & & rs_1s_2\cdots s_{n-2}c \end{bmatrix}$$

<u>Proposition</u>: $J_n(r) = r^{n-1}\cdot s_1^{n-2}\cdot s_2^{n-3}\cdots s_{n-2}^{1}$

<u>Proof.</u> By induction.

$$J_2(r) = \det\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix} = r(c^2+s^2) = r$$

$$J_n(r) = c_1 \det\begin{bmatrix} rc_1c_2 & -rs_1s_2 & 0 & \cdots & & 0 \\ rc_1s_2c_3 & rs_1s_2c_3 & -rs_1s_2s_3 & 0 & & 0 \\ \vdots & & & & & \end{bmatrix}$$

$$+ rs_1 \det\begin{bmatrix} s_1c_2 & -rs_1s_2 & 0 & \cdots & & 0 \\ s_1s_2c_3 & rs_1c_2c_3 & -rs_1s_2s_3 & 0 & \cdots & 0 \\ \vdots & & & & & \end{bmatrix}$$

$$= c_1^2 r^{n-1} s_1^{n-2} J_{n-1}(1) + r^{n-1} s_1^{n} J_{n-1}(1)$$

$$= r^{n-1} s_1^{n-2} (c_1^2 + s_1^2) J_{n-1}(1)$$

$$= r^{n-1} s_1^{n-2} J_{n-1}(1)$$