# Math 335 Sample Problems

One notebook-sized page of notes (both sides may be used) will be allowed on the final exam. The final will be comprehensive.

1. Prove that
$$\lim_{n\to\infty} \int_0^\pi \frac{\sin(nx)}{x} dx = \frac{\pi}{2}.$$

2. Define a function $\log_p(x)$ inductively by the formulas $\log_0(x) = x, \log_{p+1}(x) = \log(\log_p(x))$. Prove by induction that the series
$$\sum_{n=m}^{\infty} \frac{1}{\log_0(n)\log_1(n)\log_2(n)\ldots\log_p(n)}$$
(where $m$ is large enough for the denominators to be defined as real numbers) diverges for every $p$.

3. Suppose that $a_n > 0$, that $a_n$ is decreasing, and that $\sum_1^\infty a_n$ converges. Is it true that $\lim_{n\to\infty} na_n = 0$? If true prove it, if false give a counterexample.

4. Show that the series $\sum_1^\infty \frac{\sin nx}{\sqrt{n}}$ converges for all $x$ and uniformly on any interval of the form $[\delta, 2\pi - \delta]$, where $\delta > 0$ is small. Show that the series is not the Fourier series of a Riemann integrable function.

5. Find the solution of $u_t = 3u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \cos x \sin 5x$. (This is easier than it looks.)

6. (a) Let $\sum_{0}^{\infty} a_n x^n$ be a series with radius of convergence $R$. Substitute $re^{i\theta}$ for $x$ and get a new series involving $e^{in\theta}$. If $0 < r < R$ prove that this is a Fourier series (the variable is $\theta$).

   (b) Prove that $\sum_{0}^{\infty} r^{2n}|a_n|^2$ converges for $0 \le r < R$.

7. Folland, §8.6: problem 4.

8. Folland, §8.6: problem 10.

9. We essentially proved in class that $\displaystyle\sum_{n\neq 0} \frac{e^{inx}}{n}$ is the Fourier series of a Riemann integrable function. Since $\displaystyle\sum_{n>0} \frac{1}{n^2} < \infty$, the Riesz-Fischer Theorem asserts that $\displaystyle\sum_{n>0} \frac{e^{inx}}{n}$ is also the Fourier series of a function in $L^2[-\pi, \pi]$. Prove that $\displaystyle\sum_{n>0} \frac{e^{inx}}{n}$ is *not* the Fourier series of a piecewise continuous function. (Even more is true: this series is not the Fourier series of any Riemann integrable function.) You may use the fact that $-\log(1-z) = \displaystyle\sum_{n>0} \frac{z^n}{n}$, $|z| < 1$ and that the real part of $\log(1-z) = \log(|1-z|)$.

10. Find the function (it's a polynomial of degree 2) represented by the series $\displaystyle\sum_{k\in\mathbf{Z}, k\neq 0} \frac{e^{ikx}}{k^2}$ by using the Fourier series for the $2\pi$-periodic function equal to $x$ on $(0, 2\pi)$. You may use $\displaystyle\sum_{k\in\mathbf{Z}, k\neq 0} \frac{1}{k^2} = \frac{\pi^2}{3}$.

11. Let $f$ be a $2\pi$-periodic function and let $a$ be a fixed real number and let a new function $g$ be defined by $g(x) = f(x-a)$. What is the relation between the Fourier coefficients $\widehat{f}(n)$ and $\widehat{g}(n)$?

12. Let $f$ be a $2\pi$-periodic, piecewise smooth function. Let $\widehat{f}(n)$ be the complex Fourier coefficients of $f$. Show that there is a constant $M$ (which will depend on $f$) such that $|\widehat{f}(n)| < M/n$ for all $n$. Do **not** assume $f$ is continuous.

13. Let $f$ and $g$ be continuous $2\pi$-periodic functions. Define the *convolution* of $f$ and $g$ to be the function. $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)dt$.

    (a) Prove that $f * g$ is $2\pi$-periodic.

    (b) Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$, so the Fourier series of $f * g$ is $\sum_{-\infty}^{\infty} c_n d_n e^{inx}$, where $c_n = \widehat{f}(n),\ d_n = \widehat{g}(n)$.

14. (a) Find the cosine series of $f$ where $f(x) = 0, 0 < x < \pi/2;\ f(x) = 1, \pi/2 < x < \pi$.

    (b) Prove that the series converges for all $x$.

    (c) For which $x$ does the series converge absolutely?

15. Find the Fourier series of

$$\frac{1-r^2}{1-2r\cos x + r^2}$$

where $0 \leq r < 1$. (You don't need to integrate.)

16. let $f$ be $2\pi$-periodic, continuous, and piecewise smooth. Let $m$ be any positive integer and define the function $f_m$ by the formula $f_m(x) = f(mx)$. Prove that $\widehat{f_m}(n) = \widehat{f}\left(\frac{n}{m}\right)$ if $m$ divides $n$ and is 0 otherwise.

17. Determine $a, b, c$ so that $f_0(x) = 1, f_1(x) = x + a, f_2(x) = x^2 + bx + c$ is an orthogonal set using the inner product $< f, g >= \int_0^2 fg$ on $[0, 2]$.

18. There may be problems, statements of theorems from the text, or examples from class on the exam.