Math 335 Sample Problems

One notebook-sized page of notes (both sides may be used) will be allowed on the final exam. The final will be comprehensive.

1. Prove that
\[ \lim_{n \to \infty} \int_0^\pi \frac{\sin(nx)}{x} \, dx = \frac{\pi}{2}. \]

2. Define a function \( \log_p(x) \) inductively by the formulas \( \log_0(x) = x \), \( \log_{p+1}(x) = \log_p(\log_p(x)) \). Prove by induction that the series
\[ \sum_{n=m}^{\infty} \frac{1}{\log_0(n) \log_1(n) \log_2(n) \cdots \log_p(n)} \]
(where \( m \) is large enough for the denominators to be defined as real numbers) diverges for every \( p \).

3. Suppose that \( a_n > 0 \), that \( a_n \) is decreasing, and that \( \sum_{n=1}^{\infty} a_n \) converges.
Is it true that \( \lim_{n \to \infty} na_n = 0 \)? If true prove it, if false give a counterexample.

4. Show that the series \( \sum_{n=1}^{\infty} \frac{\sin(nx)}{\sqrt{n}} \) converges for all \( x \) and uniformly on any interval of the form \([\delta, 2\pi - \delta]\), where \( \delta > 0 \) is small. Show that the series is not the Fourier series of a Riemann integrable function.

5. Find the solution of \( u_t = 3u_{xx}, \ u(0, t) = u(\pi, t) = 0, \ u(x, 0) = \cos x \sin 5x \). (This is easier than it looks.)
6. (a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a series with radius of convergence $R$. Substitute $r e^{i\theta}$ for $x$ and get a new series involving $e^{in\theta}$. If $0 < r < R$ prove that this is a Fourier series (the variable is $\theta$).

(b) Prove that $\sum_{n=0}^{\infty} r^{2n} |a_n|^2$ converges for $0 \leq r < R$.


9. We essentially proved in class that $\sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of a Riemann integrable function. Since $\sum_{n>0} \frac{1}{n^2} < \infty$, the Riesz-Fischer Theorem asserts that $\sum_{n>0} \frac{e^{inx}}{n}$ is also the Fourier series of a function in $L^2[-\pi, \pi]$. Prove that $\sum_{n>0} \frac{e^{inx}}{n}$ is not the Fourier series of a piecewise continuous function. (Even more is true: this series is not the Fourier series of any Riemann integrable function.) You may use the fact that $-\log(1 - z) = \sum_{n>0} \frac{z^n}{n}, |z| < 1$ and that the real part of $\log(1 - z) = \log(|1 - z|).

10. Find the function (it’s a polynomial of degree 2) represented by the series $\sum_{k \in \mathbb{Z}, k \neq 0} \frac{e^{ikx}}{k^2}$ by using the Fourier series for the $2\pi$-periodic function equal to $x$ on $(0, 2\pi)$. You may use $\sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{k^2} = \frac{\pi^2}{3}$.

11. Let $f$ be a $2\pi$-periodic function and let $a$ be a fixed real number and let a new function $g$ be defined by $g(x) = f(x - a)$. What is the relation between the Fourier coefficients $\hat{f}(n)$ and $\hat{g}(n)$?
12. Let $f$ be a $2\pi$-periodic, piecewise smooth function. Let $\hat{f}(n)$ be the complex Fourier coefficients of $f$. Show that there is a constant $M$ (which will depend on $f$) such that $|\hat{f}(n)| < M/n$ for all $n$. Do \textbf{not} assume $f$ is continuous.

13. Let $f$ and $g$ be continuous $2\pi$-periodic functions. Define the \textit{convolution} of $f$ and $g$ to be the function. $f \ast g(x) = \frac{1}{2\pi} \int_{0}^{2\pi} f(x-t)g(t)dt$.

(a) Prove that $f \ast g$ is $2\pi$-periodic.

(b) Prove that $\hat{f} \ast \hat{g}(n) = \hat{f}(n)\hat{g}(n)$, so the Fourier series of $f \ast g$ is $\sum_{n=-\infty}^{\infty} c_n d_n e^{inx}$, where $c_n = \hat{f}(n)$, $d_n = \hat{g}(n)$.

14. (a) Find the cosine series of $f$ where $f(x) = 0, 0 < x < \pi/2$; $f(x) = 1, \pi/2 < x < \pi$.

(b) Prove that the series converges for all $x$.

(c) For which $x$ does the series converge absolutely?

15. Find the Fourier series of $\frac{1 - r^2}{1 - 2r \cos x + r^2}$ where $0 \leq r < 1$. (You don’t need to integrate.)

16. Let $f$ be $2\pi$-periodic, continuous, and piecewise smooth. Let $m$ be any positive integer and define the function $f_m$ by the formula $f_m(x) = f(mx)$. Prove that $\hat{f}_m(n) = \hat{f}\left(\frac{n}{m}\right)$ if $m$ divides $n$ and is 0 otherwise.

17. Determine $a, b, c$ so that $f_0(x) = 1, f_1(x) = x + a, f_2(x) = x^2 + bx + c$ is an orthogonal set using the inner product $\langle f, g \rangle = \int_{0}^{\pi} fg$ on $[0, 2]$.

18. There may be problems, statements of theorems from the text, or examples from class on the exam.