Math 335 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through §7.6

1. Let be a sequence of continuous functions in $I = [a, b]$ and suppose $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in I$. Suppose $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in I$ (point-wise convergence to 0). Is the convergence uniform? Give a proof or a counterexample.

2. Let f_n be a sequence of Riemann integrable functions on interval $I = [a, b]$. Suppose f_n converges uniformly to a limit f on I. Prove that f is Riemann integrable.

3. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1+|x|)^n}$ converges for all x, but the convergence is not uniform.

4. Assume $p \geq 1,\ q \geq 1$. Prove that
$$\int_0^1 \frac{t^{p-1}}{1+t^q}\,dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} \cdots .$$
Give careful justification of any manipulations.

5. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Does $\displaystyle\sum_{1}^{\infty} (-1)^n b_n$ converge? Give a proof or a counterexample.

6. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges uniformly for $x \in [a, b], 0 < a < b < 2\pi$, but does not converge absolutely for any x.

7. Prove that
$$\int_0^1 \left(\frac{\log(1/t)}{t}\right)^{1/2} dt = \sqrt{2\pi}.$$

8. Prove that $\displaystyle\sum_{1}^{\infty} (-1)^n \frac{\sin nx}{n}$ converges uniformly on $\{|x| < 1\}$ to a continuous function.

9. Folland §7.5, #9.

10. Let f_n be a sequence of functions defined on the open interval (a, b). Suppose $\displaystyle\lim_{x \to a^+} f_n(x) = a_n$ for all n. Suppose $\displaystyle\sum_{1}^{\infty} f_n$ converges uniformly on (a, b) to a function f. Prove that $\displaystyle\sum_{1}^{\infty} a_n$ converges and $\displaystyle\lim_{x \to a^+} f(x) = \sum_{1}^{\infty} a_n$. Do not assume f_n is continuous on (a, b).

11. Folland, §7.5, #14.

12. Suppose the series $\sum_1^\infty a_n$ converges. Prove that $\sum_1^\infty \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_1^\infty \frac{a_n}{n^x}$. Prove that $\lim_{x \to 0^+} f(x) = \sum_1^\infty a_n$.

13. You will need to know the definitions of the following terms and statements of the following theorems.

    (a) Absolute and conditional convergence of a series
    (b) Dirichlet's test
    (c) Abel's test and theorem

(d) Uniform convergence of a sequence or series of functions
(e) Weierstrass M-test
(f) Continuity of a uniform limit of continuous functions
(g) Integration and differentiation of a sequence or series
(h) Power series
(i) Radius of convergence of a power series
(j) Integration and differentiation of a power series
(k) Improper integrals dependent on a parameter
(l) Uniform convergence of an improper integral
(m) Integration and differentiation of an improper integral
(n) Gamma function

14. There may be homework problems or example problems from the text on the midterm.