Poincaré Lemma

January 18, 2011

There are several proofs of the Poincaré lemma. This note is an exposition of a proof that appears in [1]. It is more than a proof. It is an algorithm for finding a form \( \mu \) so that \( \omega = d\mu \) when \( d\omega = 0 \) on a rectangular parallelepiped in \( \mathbb{R}^n \).

We need a little information about differential forms. This discussion is brief, but should be enough to define notation. If \( x_1, x_2, \ldots, x_n \) are coordinates in \( \mathbb{R}^n \), differential forms are (ordered) objects such as \( adx_2 dx_3 dx_5 \) which satisfy such rules as

\[
adx_2 dx_3 dx_5 = -adx_2 dx_5 dx_3.
\]

In other words interchanging adjacent terms changes signs, so

\[
adx_2 dx_3 dx_5 = adx_5 dx_2 dx_3, \quad dx_2 dx_2 = 0
\]

In this notation, \( a \) is a differentiable function. The crucial operator is the exterior derivative. First it is defined for functions:

\[
df = f x_1 dx_1 + f x_2 dx_2 + \ldots f x_n dx_n,
\]

then individual terms:

\[
d(adx_I) = dadx_I,
\]

where \( I \) is a multi-index,

\[
I = (i_1, i_2, \ldots, i_k), \quad dx_I = dx_{i_1} dx_{i_2} \ldots dx_{i_k},
\]

and extended linearly. A form \( \omega \) is closed if \( d\omega = 0 \) and exact if \( \omega = d\mu \). Then since \( d^2 = 0 \), all exact forms are closed. The Poincaré Lemma is

**Lemma 1.** *On a rectangular parallelepiped, \( \Pi \), all closed forms are exact.*

**Proof.** This note will present the proof for a somewhat complicated special case. The general proof goes by induction. It should be clear from the steps in the proof that the proof is constructive. The proof will be given for a closed 2-form, \( \omega \), in 4-space. Let

\[
\omega = \alpha_1 + Pdxdw + Qdydw + Rdzdw,
\]

where \( \alpha_1 \) does not involve \( dw \). On \( \Pi \) we can find functions \( p, q, r \) so that \( p_w = P, q_w = Q, r_w = R \) (what’s the reason for this?). Now let \( \beta_1 = pdx + qdy + rdz \), and let \( \omega_1 = \omega + d\beta_1 \). Then \( \omega + d\beta_1 \) does not have any terms involving \( dw \) and \( d\omega_1 = d\omega + d^2\beta_1 = 0 \). Let \( \omega_1 = Adxdy + Bdxdz + Cdydz \). Since \( d\omega_1 = 0 \) it follows that \( B_w = 0, C_w = 0 \). Now since we are on a convex domain, \( B \) and \( C \) do not depend on \( w \) (homework exercise 2.4, # 2). Let \( B = b_z, C = c_z \) and \( \beta_2 = bdx + cdy \). Then \( \omega_2 = \omega_1 + d\beta_2 = Fdxdy \) has no terms involving \( dw \) or \( dz \). Also \( d\omega_2 = 0 \). This last equation implies that \( F_z = 0, F_w = 0 \). Let \( f_y = F \), where \( f, F \) depend only on \( x, y \). Let \( \beta_3 = f dx \) and notice that \( \omega_2 + d\beta_2 = 0 \). Writing this out we see

\[
\omega + d\beta_1 + d\beta_2 + d\beta_3 = 0.
\]
Hence
\[ \omega = d(-\beta_1 - \beta_2 - \beta_3). \]

References