THE DIFFERENTIABILITY OF THE RIEMANN FUNCTION AT CERTAIN RATIONAL MULTIPLES OF π.

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1. Introduction. Riemann is reported to have stated [1], [4], but never proved, that the continuous function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(k^2x)}{k^2}$$

is differentiable nowhere. Kahane [3] renewed the interest in this classical problem in connection with lacunary series, and refers to Weierstrass [4], who had attempted to prove Riemann's statement, did not succeed, and was led to his series representing a continuous function nowhere differentiable. To quote from Weierstrass: "Erst Riemann hat, wie ich von einigen seiner Zuhörer erfahren habe, mit Bestimmtheit ausgesprochen (i. J. 1861, oder vielleicht schon fruher), dass jene Annahme unzalässig sei, und z. B. bie der durch die unendliche Reihe

$$\sum_{n-1}^{\infty} \frac{\sin(n^2x)}{n^2}$$

dargestellten Function sich nicht bewahrheite. Leider ist der Beweis hierfur von Riemann nicht veroffenlicht worden, und scheint sich auch nicht in seinen Papieren oder mündlich Uberlieferung erhalten zu haben. Dieses ist um so mehr zu bedauern, als ich nicht einmal mit Sicherheit habe erfahren können, wie Riemann seinen Zuhörern gegenuber sich augedrückt hat."

Riemann's assertion was partially confirmed by Hardy [2], who proved that the function has no finite derivative at any point $\xi\pi$, where ξ is:

1) irrational;
2) rational of the form $2A/(4B+1)$ where A, B are integers:
3) rational of the form $(2A+1)/2(2B+1)$.

In this paper we will prove that Riemann's assertion is false, by proving the following theorem.

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THEOREM 1. *The derivative of*

$$\sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}$$

exists and is equal to $-1/2$ *at any point* $\xi\pi$, *where* ξ *is a rational number of the form* $(2A+1)/(2B+1)$, *i.e. a rational number whose numerator and denominator are odd.*

We will also extend Hardy's results by proving the following theorem.

THEOREM 2. *The derivative of the Riemann function does not exist at any point* $\xi\pi$, *where* ξ *is a rational number of the form* $(2A+1)/2^N$, *where* N *is an integer* $\geqq 1$.

In order to prove these theorems, we need two lemmas.

LEMMA 1. *Let* μ, ν *and* λ *be any integers such that*

$$0 < \mu < \nu \leqq \lambda,$$

and let τ *be any real number such that either* $-\pi/2 \leqq \tau \leqq -\pi/\lambda$, *or* $0 \leqq \tau \leqq \pi/2$. *Then*

$$\sum_{k=0}^{\infty} \left(\frac{\sin[(\lambda k+\mu)^2 x+\tau]}{(\lambda k+\mu)^2} - \frac{\sin[(\lambda k+\nu)^2 x+\tau]}{(\lambda k+\nu)^2} \right)$$

has a right derivative of $\left(\frac{\nu-\mu}{\lambda}\right)\cos\tau$ *at* 0.

LEMMA 2. *Let* μ *and* λ *be any integers such that* $0 < \mu \leqq \lambda$, *and let* τ *be any real number such that* $0 \leqq \tau < 2\pi$. *Let*

$$f(x) = \sum_{k=0}^{\infty} \frac{\sin[(\lambda k+\mu)^2 x+\tau]}{(\lambda k+\mu)^2}.$$

Then at $x=0$, *we have:*

$$\begin{aligned}
0 \leqq \tau < \pi/2 &\Rightarrow \textit{left derivative of } f \textit{ is } +\infty,\\
\pi/2 < \tau \leqq \pi &\Rightarrow \textit{right derivative of } f \textit{ is } -\infty,\\
\pi \leqq \tau < 3\pi/2 &\Rightarrow \textit{left derivative of } f \textit{ is } -\infty,\\
3\pi/2 < \tau < 2\pi &\Rightarrow \textit{right derivative of } f \textit{ is } +\infty.
\end{aligned}$$

In general, the differentiation of the Riemann function at a point P/Q where P, Q are integers, involves differentiating each subseries formed by taking the summation over those values of k in the same congruence class modulo Q. It is not difficult to prove that for any Q, these subseries are all of the form of the functions in Lemmas 1 or 2, if the coordinate system is shifted along the x-axis so that $P\pi/Q$ becomes 0. If no more than one of these subseries is of the type in Lemma 2, then it is obviously possible to find the derivative of the entire series by adding the derivatives of all the subseries. If there is more than one subseries of the type in Lemma 2, this is not generally possible, either because one ends up with both unknown right and left derivatives, or because one must add derivatives of $+\infty$ and $-\infty$. In particular, this is true of rational multiples of π, of the form $2A/(4B+3)$ and $(2A+1)/2^N(2B+1)$ where $N \geqq 2$. Solutions for these points will have to await a better approximation of the values of the Lemma 2 type series near 0 than are provided by Lemma 2.

Note that Lemmas 1 and 2 simply state in somewhat more generalized form, that the Riemann function has a derivative of $-1/2$ at π and $+\infty$ at 0. Therefore, before proving Lemmas 1 and 2, we will briefly outline the proofs of these simpler results, which in fact, formed the basis for the rest of the paper historically.

Let f be the Riemann function, that is

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}.$$

To prove the differentiability at π, fix n. We then show that

$$\left| f(x) - \frac{x-\pi}{2} \right| < \frac{c(x-\pi)}{n}$$

with a suitable constant c, for x sufficiently close to π. Indeed, take

$$| x - \pi | < \frac{\pi}{2n^{14}},$$

and partition the series atthose integers k closest to

$$\frac{1}{n}\sqrt{\frac{\pi}{2(x-\pi)}} \quad \text{and} \quad \frac{\pi}{2n(x-\pi)}.$$

Then one shows that the first part approaches 0, the second part approaches $(x-\pi)/2$, and the tail end approaches 0.

On the other hand, to show that the derivative of f at 0 is $+\infty$, we fix n, and then show that $f(x) > nx$ for sufficiently small x. For this, let $| x | < \pi/2cn^2$, where c is a suitable constant. Partition the series after $n+1$ at

$$\sqrt{\frac{\pi}{2x}} \quad \text{and} \quad \sqrt{\frac{\pi}{x}}.$$

We then show that the first part is $> nx$, the third is > 0, and the second part is greater than the absolute value of the tail end.

2. Preliminary remarks on Lemma 1. It will be convenient, in proving Lemma 1, to adjust each term of the series so that it passes through the origin. This can be done by altering the series to read

$$\sum_{k=0}^{\infty}\left(\frac{\sin[(\lambda k+\mu)^2x+\tau]-\sin\tau}{(\lambda k+\mu)^2}-\frac{\sin[(\lambda k+\nu)^2x+\tau]-\sin\tau}{(\lambda k+\nu)^2}\right)$$

Since $\dfrac{\sin\tau}{(\lambda k+\mu)^2}$ and $\dfrac{\sin\tau}{(\lambda k+\nu)^2}$ are constants, this modification does not affect the existence and value of the derivative.

Since for most of this proof we will be examining expressions of the form $\dfrac{\sin[(\lambda k+\mu)^2x+\tau]-\sin\tau}{(\lambda k+\mu)^2}$ as functions of k, keeping x constant, we will, for convenience, write expressions of the form $\dfrac{\sin(i^2x+\tau)-\sin\tau}{i^2}$ as $S(i)$.

We will prove the following form of Lemma 1:

For $n>\lambda$, $0<x<\dfrac{2\pi}{\lambda^2n^{14}}$, we have:

$$\left|\sum_{k=0}^{\infty}[S(\lambda k+\mu)-S(\lambda k+\nu)]-x\left(\frac{\nu-\mu}{\lambda}\right)\cos\tau\right|<\frac{469x}{n}.$$

First we divide the series into two parts, the sum over all $k<b$ and the sum over all $k\geqq b$, where b is the least integer greater than $\dfrac{\pi}{n\lambda^2x}$. Now we will prove that for all x such that $0<x<\dfrac{2\pi}{\lambda^2n^{14}}$, we have

$$\left|\sum_{k=0}^{b-1}[S(\lambda k+\mu)-S(\lambda k+\nu)]-x\left(\frac{\nu-\mu}{\lambda}\right)\cos\tau\right|<\frac{125\pi x}{n}$$

and

$$\left|\sum_{k=b}^{\infty}[S(\lambda k+\mu)-S(\lambda k+\nu)]\right|<\frac{76x}{n},\ \text{for } n>\lambda.$$

3. Values of k less than b. Basically, we will show that for all k less than b, $S(\lambda k+\mu)-S(\lambda k+\nu)$ is very close to

$$\left(\frac{\nu-\mu}{\lambda}\right)[S(\lambda k+\mu)-S(\lambda(k+1)+\mu)].$$

This means that $\sum_{k=0}^{b-1}[S(\lambda k+\mu)-S(\lambda k+\nu)]$ is very close to

$$\sum_{k=0}^{b-1}[S(\lambda k+\mu)-S(\lambda(k+1)+\mu)](\frac{\nu-\mu}{\lambda}),$$

which is equal to $(\frac{\nu-\mu}{\lambda})[S(\mu)-S(\lambda b+\mu)]$, since all the middle terms cancel out. Since $x<\frac{2\pi}{\lambda^2 n^{14}}<<\frac{1}{\mu^2}$, $S(\mu)$ is approximately $x\cos\tau$, and, since

$$\frac{1}{(\lambda b+\mu)^2}<\frac{1}{\lambda^2 b^2}<\frac{n^2\lambda^2 x^2}{\pi^2}<\frac{2x}{\pi n^{12}}<<x,\ S(\lambda b+\mu)<<x,$$

so $\sum_{k=0}^{b-1}[S(\lambda k+\mu)-S(\lambda k+\nu)]$ is very close to $x(\frac{\nu-\mu}{\lambda})\cos\tau$.

First consider $k<c$, where c is the least integer greater than $\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}$. For all positive integers i, $\frac{dS(i)}{dx}=\cos(\tau+i^2x)$. Consider two positive integers i and j such that $i<j\leqq\lambda c$. Now $c<\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}+1<\frac{2}{\lambda n}\sqrt{\frac{\pi}{2x}}$, since $x<\frac{2\pi}{\lambda^2 n^{14}}$. Therefore $(\lambda c)^2<\frac{2\pi}{n^2 x}<\frac{\pi}{\lambda x}$, and $i^2x<j^2x<\frac{\pi}{\lambda}$. Therefore

$$\frac{-\pi}{2}\leqq\tau\leqq\frac{-\pi}{\lambda}\Rightarrow\cos(\tau+i^2x)<\cos(\tau+j^2x)$$

and

$$0\leqq\tau\leqq\frac{\pi}{2}\Rightarrow\cos(\tau+i^2x)>\cos(\tau+j^2x),$$

for all x such that $0<x<\frac{2\pi}{\lambda^2 n^{14}}$. Since $S(i)=S(j)$ and $\frac{dS(i)}{dx}=\frac{dS(j)}{dx}$ at $x=0$, $\frac{-\pi}{2}\leqq\tau\leqq\frac{-\pi}{\lambda}\Rightarrow S(i)<S(j)$ and $0\leqq\tau\leqq\frac{\pi}{2}\Rightarrow S(i)>S(j)$ for all x such that $0<x<\frac{2\pi}{\lambda^2 n^{14}}$. Therefore, either $i<j\Rightarrow S(i)<S(j)$ for all $i,j\leqq\lambda c$ or $i<j\Rightarrow S(i)>S(j)$ for all $i,j\leqq\lambda c$. Therefore

$$\begin{aligned}|\sum_{k=0}^{c-1}[S(\lambda k+\mu)-S(\lambda k+\nu)]|&<|S(\mu)-S(\lambda c+\mu)|\\&<|x\cos\tau-S(\lambda c+\mu)|<|x[\cos\tau-\cos(\tau+(\lambda c+\mu)^2x)]|\\&<(\lambda c+\mu)^2x^2<\frac{2\pi x}{n^2}.\end{aligned}$$

All $k<c$ can therefore be disregarded in calculating the value of $\sum_{k=0}^{b-1}[S(\lambda k+\mu)-S(\lambda k+\nu)]$.

Now we will consider values of k from c to $b-1$, but first we need the following lemma:

LEMMA 3. *For all integers* $i > 4$,

$$|S(i) - 2S(i+1) + S(i+2)| < 11ix^3 + 16x^2 + \frac{6x}{i^2} + \frac{16}{i^4}.$$

Proof

$$\frac{\sin[(i+1)^2x+\tau] - \sin\tau}{i^2} - \frac{\sin[(i+1)^2x+\tau] - \sin\tau}{(i+1)^2}$$
$$= \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2} - \frac{(2i+1)\sin\tau}{i^2(i+1)^2}.$$

$$S(i) - S(i+1) = \frac{\sin(i^2x+\tau) - \sin\tau}{i^2} - \frac{\sin[(i+1)^2x+\tau] - \sin\tau}{i^2}$$
$$- \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2} + \frac{(2i+1)\sin\tau}{i^2(i+1)^2}$$
$$= \frac{\sin(i^2x+\tau)}{i^2} - \frac{\sin[(i+1)^2x+\tau]}{i^2} - \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2}$$
$$+ \frac{(2i+1)\sin\tau}{i^2(i+1)^2}.$$

For all x, say $x = x_1$, $\dfrac{\sin(i^2x+\tau)}{i^2} - \dfrac{\sin[(i+1)^2x+\tau]}{i^2} = -sr_1$ where $s = \dfrac{(i+1)^2x_1}{i^2} - x_1 = \dfrac{(2i+1)x_1}{i^2}$, and r_1 is the derivative of $\dfrac{\sin(i^2x+\tau)}{i^2}$ for some x between x_1 and x_1+s. Let $f(x) = \dfrac{\sin(i^2x+\tau)}{i^2}$ and $g(x) = \dfrac{\sin[(i+1)^2x+\tau]}{(i+1)^2}$. For any x, r can be approximated as $f'(x)$: $f''(x) = -i^2\sin(i^2x+\tau)$, $\max|f''(x)| = i^2$, $|r - f'(x)| \leqq i^2s = x(2i+1)$,

$$\left|\frac{\sin(i^2x+\tau)}{i^2} - \frac{\sin[(i+1)^2x+\tau]}{i^2} + \frac{f'(x)(2i+1)x}{i^2}\right|$$
$$< \frac{(2i+1)^2x^2}{i^2} < 5x^2 \text{ (for } i > 4).$$

We have Inequality 1:

$$\Big|S(i) - S(i+1) - \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2}$$
$$+ \frac{(2i+1)\sin\tau}{i^2(i+1)^2} + \frac{f'(x)(2i+1)x}{i^2}\Big| < 5x^2$$

Similarly, Inequality 2:

$$\left| S(i+1) - S(i+2) - \frac{(2i+3)\sin[(i+2)^2x+\tau]}{(i+1)^2(i+2)^2} + \frac{(2i+3)\sin\tau}{(i+1)^2(i+2)^2} + \frac{g'(x)(2i+3)x}{(i+1)^2} \right| < 5x^2.$$

Let $h(x) = \dfrac{\cos(i^2x+\tau)}{i^2}$.

Then

$$\left| \frac{\cos(i^2x+\tau)}{i^2} - \frac{\cos[(i+1)^2x+\tau]}{i^2} + \frac{h'(x)(2i+1)x}{i^2} \right| < 5x^2$$

and

$$| \cos(i^2x+\tau) - \cos[(i+1)^2x+\tau] | < |h'(x)|(2i+1)x + 5i^2x^2$$
$$< (2i+1)x + 5i^2x^2.$$
$$|f'(x) - g'(x)| < (2i+1)x + 5i^2x^2.$$

$$\frac{g'(x)(2i+3)x}{(i+1)^2} - \frac{g'(x)(2i+1)x}{i^2} = \frac{2g'(x)x}{(i+1)^2} - \frac{(2i+1)^2g'(x)x}{i^2(i+1)^2}$$
$$> \frac{-2g'(x)x}{i^2}.$$

$$\left| \frac{g'(x)(2i+3)x}{(i+1)^2} - \frac{g'(x)(2i+1)x}{i^2} \right| < \frac{2x}{i^2}$$

$$\left| \frac{g'(x)(2i+1)x}{i^2} - \frac{f'(x)(2i+1)x}{i^2} \right| < \frac{(2i+1)^2x^2}{i^2} + 5(2i+1)x^3.$$

Inequality 3:

$$\left| \frac{g'(x)(2i+3)x}{(i+1)^2} - \frac{f'(x)(2i+1)x}{i^2} \right| < \frac{2x}{i^2} + 5x^2 + 5(2i+1)x^3$$

$$\frac{(2i+1)\sin\tau}{i^2(i+1)^2} - \frac{(2i+3)\sin\tau}{(i+1)^2(i+2)^2} = \frac{[(2i+1)(i+2)^2 - (2i+3)i^2]\sin\tau}{i^2(i+1)^2(i+2)^2}$$
$$= \frac{[6i^2+12i+4]\sin\tau}{i^2(i+1)^2(i+2)^2}$$

$$\left| \frac{(6i^2+12i+4)\sin\tau}{i^2(i+1)^2(i+2)^2} \right| < \left| \frac{(6i^2+12i+4)\sin\tau}{i^6} \right| < \frac{10.}{i^4}.$$

Inequality 4:

$$\left| \frac{(2i+1)\sin\tau}{i^2(i+1)^2} - \frac{(2i+3)\sin\tau}{(i+1)^2(i+2)^2} \right| < \frac{10}{i^4}.$$

$$\left| \frac{(2i+3)\sin[(i+2)^2x+\tau]}{(i+1)^2(i+2)^2} - \frac{(2i+1)\sin[(i+2)^2x+\tau]}{i^2(i+1)^2} \right|$$

$$= \left| \frac{-(4i+4)(2i+3)\sin[(i+2)^2x+\tau]}{i^2(i+1)^2(i+2)^2} + \frac{2\sin[(i+2)^2x+\tau]}{i^2(i+1)^2} \right|$$

$$< \left| \frac{6\sin[(i+2)^2x+\tau]}{i^4} \right| < \frac{6}{i^4}.$$

$$\left| \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2} - \frac{(2i+1)\sin[(i+2)^2x+\tau]}{i^2(i+1)^2} \right|$$

$$< \frac{g'(x)(2i+3)(2i+1)x}{i^2(i+1)^2} + \frac{(2i+3)^2(2i+1)x^2}{i^2(i+1)^2} < \frac{4x}{i^2} + \frac{10x^2}{i}.$$

Inequality 5:

$$\left| \frac{(2i+3)\sin[(i+2)^2x+\tau]}{(i+1)^2(i+2)^2} - \frac{(2i+1)\sin[(i+1)^2x+\tau]}{i^2(i+1)^2} \right|$$

$$< \frac{6}{i^4} + \frac{4x}{i^2} + \frac{10x^2}{i}.$$

Adding inequalities 1 and 2, and substituting inequalities 3, 4, and 5, we get:

$$|S(i) - 2S(i+1) + S(i+2)| < 10x^2 + \frac{2x}{i^2} + 5x^2 + 5(2i+1)x^3$$

$$+ \frac{10}{i^4} + \frac{6}{i^4} + \frac{4x}{i^2} + \frac{10x^2}{i} < 11ix^3 + 16x^2 + \frac{6x}{i^2} + \frac{16}{i^4},$$

which is Lemma 3.

COROLLARY 3a. *For all* $k > 2$,

$$|[S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)]|$$

$$< 4\lambda^2(11\lambda k x^3 + 16x^2 + \frac{6x}{\lambda^2k^2} + \frac{16}{\lambda^4k^4}).$$

Proof. By induction, for all $j < 2\lambda$,

$$|S(\lambda k) - S(\lambda k+1) - S(\lambda k+j) + S(\lambda k+j+1)|$$

$$< 2\lambda(11\lambda k x^3 + 16x^2 + \frac{6x}{\lambda^2k^2} + \frac{16}{\lambda^4k^4}).$$

Therefore, taking the average of a sequence of terms of the form

$$S(\lambda k+j) - S(\lambda k+j+1)\,;\; j < 2\lambda,$$

$$| S(\lambda k) - S(\lambda k+1) - \frac{1}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] |$$
$$< 2\lambda(11\lambda k x^3 + 16x^2 + \frac{6x}{\lambda^2 k^2} + \frac{16}{\lambda^4 k^4}).$$

And for all $j < 2\lambda$,

$$| S(\lambda k+j) - S(\lambda k+j+1) - \frac{1}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] |$$
$$< 4\lambda(11\lambda k x^3 + 16x^2 + \frac{6x}{\lambda^2 k^2} + \frac{16}{\lambda^4 k^4}),$$

from which Corollary 3a follows immediately.

Now let a be the least integer $> \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}}$. We will consider the values of k from c to $a-1$ and from a to $b-1$ separately.

Frst from c to $a-1$:

$$c \leqq k \leqq a-1 \Rightarrow \frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}} < k \leqq \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}}$$

$$k \leqq \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}} \Rightarrow 11kx^3 \leqq 11x^3 \sqrt{\frac{\pi}{2x}} = \frac{11\pi x^2}{2}\sqrt{\frac{2x}{\pi}}$$

$$k > \frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}} \Rightarrow \frac{6x}{\lambda^2 k^2} < 6n^2 x(\frac{2x}{\pi}) = \frac{12n^2x^2}{\pi}$$

$$k > \frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}} \Rightarrow \frac{16}{\lambda^4 k^4} < 16n^4(\frac{2x}{\pi})^2 = \frac{64n^4x^2}{\pi^2}.$$

Therefore $c \leqq k \leqq a-1 \Rightarrow$

$$| [S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] |$$
$$< 4\lambda^2(\frac{11\pi x^2}{2}\sqrt{\frac{2x}{\pi}} + 16x^2 + \frac{12n^2x^2}{\pi} + \frac{64n^4x^2}{\pi^2}) < 30n^4\lambda^2x^2,$$

and

$$|\sum_{k=c}^{a-1}\{[S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)]\}|$$
$$< (a-1)30n^4\lambda^2x^2 \leqq 30n^4\lambda x^2\sqrt{\frac{\pi}{2x}} = 15\pi n^4\lambda \mathrm{x}\sqrt{\frac{\pi}{2x}}$$
$$< 15\pi n^4\lambda x(\frac{2}{\lambda n^7}) = \frac{30\pi x}{n^3}.$$

Next, the values of k from a to $b-1$:

$$a \leqq k \leqq b-1 \Rightarrow \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}} < k \leqq \frac{\pi}{n\lambda^2 x}$$

$$k \leqq \frac{\pi}{n\lambda^2 x} \Rightarrow 11\lambda k x^3 \leqq \frac{11\pi x^2}{2\lambda n}$$

$$k > \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}} \Rightarrow \frac{6x}{\lambda^2 k^2} < \frac{12x^2}{\pi}$$

$$k > \frac{1}{\lambda}\sqrt{\frac{\pi}{2x}} \Rightarrow \frac{16}{\lambda^4 k^4} < \frac{64x^2}{\pi^2}.$$

Therefore, $a \leqq k \leqq b-1 \Rightarrow$

$$\left| [S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] \right|$$
$$< 4\lambda^2\left(\frac{11\pi x^2}{2\lambda n} + 16x^2 + \frac{12x^2}{\pi} + \frac{64x^2}{\pi^2}\right) < 120\lambda^2 x^2$$

and

$$\left| \sum_{k=a}^{b-1} \{ [S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] \} \right|$$
$$< (b-1)120\lambda^2 x^2 \leqq \frac{120\pi x}{n}.$$

Combining both sums,

$$\left| \sum_{k=c}^{b-1} \{ [S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda k+\mu) - S(\lambda(k+1)+\mu)] \} \right|$$
$$< \frac{30\pi x}{n^3} + \frac{120\pi x}{n} < \frac{124\pi x}{n}.$$

Therefore,

$$\left| \sum_{k=c}^{b-1} [S(\lambda k+\mu) - S(\lambda k+\nu)] - \frac{\nu-\mu}{\lambda}[S(\lambda c+\mu) - S(\lambda b+\mu)] \right| < \frac{124\pi x}{n}.$$

And finally, since $S(\lambda c+\mu) - x\cos\tau$ is on the order of $\frac{x}{n^2}$, and $S(\lambda b+\mu) < \frac{1}{\lambda^2 b^2} < \frac{x}{n^{12}}$,

$$\left| \sum_{k=0}^{b-1} [S(\lambda k+\mu) - S(\lambda k+\nu)] - \left(\frac{\nu-\mu}{\lambda}\right) x\cos\tau \right| < \frac{125\pi x}{n}.$$

4. Values of k greater than or equal to b. Since $S(\lambda k+\mu) - S(\lambda k+\nu)$

ranges approximately from $\frac{-2}{\lambda^2 k^2}$ to $\frac{2}{\lambda^2 k^2}$ and $\sum_{k=b}^{\infty} \frac{2}{\lambda^2 k^2} > \frac{2}{\lambda^2 b} \approx \frac{2nx}{\pi}$, if we are to prove that $\sum_{k=b}^{\infty} [S(\lambda k + \mu) - S(\lambda k + \nu)]$ is much less than x, we must show that there is a reasonably symmetric distribution of the values of $S(\lambda k + \mu) - S(\lambda k + \nu)$ within that range, so that the positive and negative values nearly cancel each other out.

Now the value of $S(\lambda k + \mu) - S(\lambda k + \nu)$ within the range from $\frac{-2}{\lambda^2 k^2}$ to $\frac{2}{\lambda^2 k^2}$ depends on the phases of $S(\lambda k + \mu)$ and $S(\lambda k + \nu)$ at x. The phase difference between $S(\lambda k + \mu)$ and $S(\lambda k + \nu)$ tends to remain constant over long sequences of consecutive integers k, but in general the phase of $S(\lambda k + \mu)$ will cover a broad range of values, evenly distributed between 0° and 360°, unless the phase difference between $S(\lambda k + \mu)$ and $S(\lambda(k+1) + \mu)$ (which also tends to remain constant over long sequences of k) is close to a fraction of 360° with a small denominator, so that every few integers the phase of $S(\lambda k + \mu)$ repeats itself.

We will define a set K consisting of those values of k for which this phase difference is sufficiently close to a fraction with a small denominator, and then show that there are not enough elements of K to significantly affect the value of $\sum_{k=b}^{\infty} [S(\lambda k + \mu) - S(\lambda k + \nu)]$.

We define K such that:

$$k \in K \Longleftrightarrow \exists \text{ integers } i, j \text{ such that } j < n^2$$

and

$$\left| k - \frac{2\pi i}{\lambda^2 x j} \right| \leqq \frac{4n}{\lambda} \sqrt{\frac{\pi}{2x}}.$$

First we will show that if Q is a subset of K, then

$$\Big| \sum_{\substack{k \in Q \\ k \geqq b}} [S(\lambda k + \mu) - S(\lambda k + \nu)] \Big| < \frac{20x}{\pi n}.$$

We partition the set of integers $\geqq b$ into an infinite number of disjoint subsets of the form S_m such that

$$k \in S_m \Longleftrightarrow \frac{\pi m}{n^4 \lambda^2 x} < k \leqq \frac{\pi(m+1)}{n^4 \lambda^2 x}.$$

Now, $\dfrac{i_1}{j_1}-\dfrac{i_2}{j_2}=\dfrac{i_1j_2-i_2j_1}{j_1j_2}$

$\dfrac{i_1}{j_1}\neq\dfrac{i_2}{j_2}\Rightarrow i_1j_2-i_2j_1\neq 0\Rightarrow |\,i_1j_2-i_2j_1\,|\geqq 1.$

Therefore, if $j_1, j_2<n^2$, then $\dfrac{i_1}{j_1}\neq\dfrac{i_2}{j_2}\Rightarrow\left|\dfrac{i_1}{j_1}-\dfrac{i_2}{j_2}\right|>\dfrac{1}{n^4}$

$$\text{and } \left|\frac{2\pi i_1}{\lambda^2xj_1}-\frac{2\pi i_2}{\lambda^2xj_2}\right|>\frac{2\pi}{n^4\lambda^2x}.$$

Therefore, for each m, there exists at most one i and j such that

$$\frac{\pi(m-\frac{1}{2})}{n^4\lambda^2x}<\frac{2\pi i}{\lambda^2xj}\leqq\frac{\pi(m+\frac{3}{2})}{n^4\lambda^2x}.$$

In other words, for each S_m, there exists at most one i and j such that

$$\exists k\in S_m \text{ such that } \left|k-\frac{2\pi i}{\lambda^2xj}\right|<\frac{\pi}{2n^4\lambda^2x}.$$

Since $0<x<\dfrac{2\pi}{\lambda^2n^{14}}$, $\dfrac{\pi}{2n^4\lambda^2x}>\dfrac{n^3}{2\lambda}\sqrt{\dfrac{\pi}{2x}}>\dfrac{4n}{\lambda}\sqrt{\dfrac{\pi}{2x}}$, so there exists at most on i and j such that

$$\exists k\in S_m \text{ such that } \left|k-\frac{2\pi i}{\lambda^2xj}\right|<\frac{4n}{\lambda}\sqrt{\frac{\pi}{2x}}.$$

Therefore there exist at most $\dfrac{8n}{\lambda}\sqrt{\dfrac{\pi}{2x}}$ elements of K in each S_m, which implies that there exists no more than $\dfrac{8n}{\lambda}\sqrt{\dfrac{\pi}{2x}}$ elements of Q in each S_m if Q is any subset of K.

$$k\in S_m\Rightarrow |\,S(\lambda k+\mu)\,| \text{ and } |\,S(\lambda k+\nu)\,| \text{ are both } <\frac{1}{\lambda^2k^2}<\frac{n^8\lambda^2x^2}{\pi^2m^2}.$$

Therefore, $\left|\sum_{k\in S_m\cap Q}[S(\lambda k+\mu)-S(\lambda k+\nu)]\right|<\left(\dfrac{16n}{\lambda}\sqrt{\dfrac{\pi}{2x}}\right)\left(\dfrac{n^8\lambda^2x^2}{\pi^2m^2}\right)$. Since all elements of S_m are $\geqq b>\dfrac{\pi}{n\lambda^2x}$, $m\geqq n^3$. Therefore

$$\left|\sum_{k\in S_m\cap Q}[S(\lambda k+\mu)-S(\lambda k+\nu)]\right|<\left(\frac{18n}{\lambda}\sqrt{\frac{\pi}{2x}}\right)\left(\frac{n^8\lambda^2x^2}{\pi^2(m+1)^2}\right)$$
$$=\left(\frac{18n}{\lambda^3}\sqrt{\frac{\pi}{2x}}\right)\left(\frac{n^4\lambda^2x}{\pi(m+1)}\right)^2.$$

Since there are more than $\dfrac{\pi}{n^4\lambda^2x}-1$ elements of S_m, all of them $<\dfrac{\pi(m+1)}{n^4\lambda^2x}$,

$$\left(\frac{18n}{\lambda^3}\sqrt{\frac{\pi}{2x}}\right)\left(\frac{n^4\lambda^2 x}{\pi(m+1)}\right)_2 < \frac{\left(\frac{18n}{\lambda^3}\sqrt{\frac{\pi}{2x}}\right)}{\left(\frac{\pi}{n^4\lambda^2 x}-1\right)}\sum_{k\in S_m}\frac{1}{k^2} < \frac{\left(\frac{19n}{\lambda^3}\sqrt{\frac{\pi}{2x}}\right)}{\left(\frac{\pi}{n^4\lambda^2 x}\right)}\sum_{k\in S_m}\frac{1}{k^2}$$

$$=\frac{19n^5}{2\lambda}\sqrt{\frac{2x}{\pi}}\sum_{k\in S_m}\frac{1}{k^2} < \frac{19}{\lambda^2 n^2}\sum_{k\in S_m}\frac{1}{k^2}.$$

Now we need a lemma, whose obvious proof we leave to the reader.

LEMMA 4. *For all integers* $N\geqq 2$, $\sum_{k=N}^{\infty}\frac{1}{k^2}<\frac{1}{N-1}$.

Combining all S_m, and applying Lemma 4,

$$\left|\sum_{\substack{k\in Q\\ k\geqq b}}[S(\lambda+\mu)-S(\lambda k+\nu)]\right| < \frac{19}{\lambda^2 n^2}\sum_{k=b}^{\infty}\frac{1}{k^2} < \frac{19}{n^2\lambda^2(b-1)} < \frac{20}{n^2\lambda^2 b}$$

$$<\frac{20x}{\pi n}.$$

Now we produce a set Q which is a subset of K and proceed to demonstrate that

$$\left|\sum_{\substack{k\notin Q\\ k\geqq b}}[S(\lambda k+\mu)-S(\lambda k+\nu)]\right| < \frac{69x}{n}.$$

We define Q as the set of all integers $\geqq b$ not belonging to one of an infinite sequence of disjoint sets of the form R_u, which together include all integers $\geqq b$ which are not elements of K, as well as some integers which are elements of K. We define R_u in the following manner: Let $t_1=$ the least integer $\geqq b$ which is not an element of K. Let $j_u=$ the least integer such that there exists an integer i such that

$$\left|j_u t_u-\frac{2\pi i}{\lambda^2 x}\right|\leqq\frac{4n}{\lambda}\sqrt{\frac{\pi}{2x}}.$$

Let $t_{u+1}=$ the least integer $\geqq t_u+j_u$ which is not an element of K. Let R_u be the sets consisting of the j_u consecutive integers starting with t_u.

Note that no t_u is an element of K, and therefore, for all u, $j_u\geqq n^2$. Notes furthermore that, for all u,

$$j_u\leqq\frac{\left(\frac{2\pi}{\lambda^2 x}\right)}{\left(\frac{4n}{\lambda}\sqrt{\frac{\pi}{2x}}\right)}=\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}.$$

Therefore, each R_u must have at least n^2 but no more than $\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}$ elements.

We will now prove that for each u,

$$\left|\sum_{k\in R_u}[S(\lambda k+\mu)-S(\lambda k+\nu)]\right|<\frac{67\pi j_u}{\lambda^2 t_u{}^2 n^2}.$$

In the following discussion, t and j will represent respectively any t_u and j_u. Let $f_z(x)=S[\lambda(t+z)]$ for any integer z. There exists a unique real number α such that $0\leqq\alpha<\frac{2\pi}{\lambda^2t^2}$ and such that

$$f'_0(x-\alpha)=1.$$

Let p be any integer such that $0\leqq p<j$. Then there exists a unique ψ_p such that $0\leqq\psi_p<\frac{2\pi}{\lambda^2(t+p)^2}\leqq\frac{2\pi}{\lambda^2t^2}$ and such that $f'_p(x-\alpha-\psi_p)=1$. Let $\beta_p=(x-\alpha-\psi_p)\left(1-\frac{(t+p)^2}{(t+p+1)^2}\right)=(x-\alpha-\psi_p)\left(\frac{2(t+p)+1}{(t+p+1)^2}\right)$. Then $f'_{p+1}(x-\alpha-\psi_p-\beta_p)=1$. Since $t+p\geqq t\geqq b>\frac{\pi}{n\lambda^2x}$,

$$\alpha<\frac{2nx}{t},\ \psi_p<\frac{2nx}{t},\text{ and }\alpha+\psi_p<\frac{4nx}{t}$$

$$(\alpha+\psi_p)\left(\frac{2(t+p)+1}{(t+p+1)^2}\right)<\frac{2(\alpha+\psi_p)}{t}<\frac{8nx}{t^2}$$

$$\left|\beta_p-x\left(\frac{2(t+p)+1}{(t+p+1)^2}\right)\right|<2\,\frac{8nx}{t^2}$$

$$\left(\frac{2(t+p)+1}{t^2}\right)-\left(\frac{2(t+p)+1}{(t+p+1)^2}\right)$$

$$=\frac{(2t+2p+1)(2tp+2t+p^2+2p+1)}{t^2(t+p+1)^2}<\frac{4(p+1)}{t^2}$$

$$\left|\beta_p-x\left(\frac{2(t+p)+1}{t^2}\right)\right|<\frac{8nx}{t^2}+\frac{4(p+1)x}{t^2}$$

$$\left|\beta_p-\frac{2x}{t}\right|<\frac{8nx}{t^2}+\frac{4(p+1)x}{t^2}+\frac{(2p+1)x}{t^2}<\frac{[8n+6(p+1)]x}{t^2}$$

$$<\frac{7x}{\lambda nt^2}\sqrt{\frac{\pi}{2x}}<\frac{7\pi}{2\lambda nt^2}\sqrt{\frac{2x}{\pi}},\ \text{since } n\ll\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}\text{ and } p<\frac{1}{\lambda n}\sqrt{\frac{\pi}{2x}}.$$

Let $\theta_p=\sum_{z=0}^{p-1}\beta_z$. Then $f'_p(x-\alpha-\theta_p)=1$. Therefore, there exists an integer N such that $\theta_p=\psi_p+\frac{2\pi N}{\lambda^2(t+p)^2}$. Now

$$|\theta_p - \frac{2px}{t}| < \frac{7\pi p}{2\lambda n t^2}\sqrt{\frac{2x}{\pi}} < \frac{7\pi}{2\lambda^2 n^2 t^2},$$

so that there exists N such that $|\psi_p + \frac{2\pi N}{\lambda^2(t+p)^2} - \frac{2px}{t}| < \frac{7\pi}{2\lambda^2 n^2 t^2}$. Now,

$$|\frac{2\pi N}{\lambda^2(t+p)^2}| < \max(\psi_p, \frac{2px}{t}) + \frac{7\pi}{2\lambda^2 n^2 t^2}$$

$$< \max(\frac{2\pi}{\lambda^2 t^2}, \frac{\pi}{\lambda n t}\sqrt{\frac{2x}{\pi}}) + \frac{7\pi}{2\lambda^2 n^2 t^2}$$

$$< \max(\frac{2\pi}{\lambda^2 t^2}, \frac{\pi}{\lambda^2 n^2 tp}) + \frac{7\pi}{2\lambda^2 n^2 t^2} < \frac{2\pi}{\lambda^2 n^2 p(t+p)},$$

since $n^2 p \ll t$, and

$$|\frac{2\pi N}{\lambda^2 t^2} - \frac{2\pi N}{\lambda^2(t+p)^2}| = |\frac{2\pi N}{\lambda^2}| \frac{2tp+p^2}{t^2(t+p)^2} < \frac{4\pi}{\lambda^2 n^2 t^2}.$$

So there exists N such that

$$|\psi_p + \frac{2\pi N}{\lambda^2 t^2} - \frac{2px}{t}| < \frac{15\pi}{2\lambda^2 n^2 t^2}$$

which implies

$$|\frac{t^2\lambda^2\psi_p}{2\pi} + N - \frac{\lambda^2 tpx}{\pi}| < \frac{4}{n^2}.$$

Now there exists i such that

$$|jt - \frac{2\pi i}{\lambda^2 x}| \leqq \frac{4n}{\lambda}\sqrt{\frac{\pi}{2x}}.$$

There exists i such that

$$|t + \frac{2\pi i}{\lambda^2 xj}| \leqq \frac{4n}{\lambda j}\sqrt{\frac{\pi}{2x}} < \frac{4}{\lambda n}\sqrt{\frac{\pi}{2x}} \text{ since } j \geqq n^2.$$

There exists i such that

$$|\frac{\lambda^2 tpx}{\pi} - \frac{2pi}{j}| \leqq \frac{2\lambda p}{n}\sqrt{\frac{2x}{\pi}} < \frac{2}{n^2}.$$

Therefore there exists i such that

$$|\frac{t^2\lambda^2\psi_\rho}{2\pi} + N - \frac{2pi}{j}| < \frac{6}{n^2}.$$

$0 \leqq \frac{t^2\lambda^2\psi_\rho}{2\pi} < 1$, since $0 \leqq \psi_\rho < \frac{2\pi}{\lambda^2 t^2}$. Therefore $\frac{-6}{n^2} < \frac{2pi}{j} - N < 1 + \frac{6}{n^2}$.

We define q_p and ω_p as follows:

Let $q_p = 2pi - Nj$ and $\omega_p = \psi_p$ if $0 \leqq \frac{2pi}{j} - N < 1$.

Let $q_p = 2pi - (N+1)j$ and $\omega_p = \psi_p - \frac{2\pi}{\lambda^2(t+p)^2}$ if $\frac{-6}{n^2} < \frac{2pi}{j} - N < 0$.

Let $q_p = 2pi - (N-1)j$ and $\omega_p = \psi_p + \frac{2\pi}{\lambda^2(t+p)^2}$ if

$$1 \leqq \frac{2pi}{j} - N < 1 + \frac{6}{n^2}.$$

Then $f'_p(x - \alpha - \omega_p) = 1$, $0 \leqq \frac{q_p}{j} < 1$, and, since

$$1 - \frac{t^2}{(t+p)^2} \ll \frac{1}{n^2}, \quad \left| \frac{t^2\lambda^2\omega_p}{2\pi} - \frac{q_p}{j} \right| < \frac{7}{n^2}.$$

Now i must be prime relative to j, since j is the least integer such that

$$\left| jt - \frac{2\pi i}{\lambda^2 x} \right| \leqq \frac{4n}{\lambda}\sqrt{\frac{\pi}{2x}}.$$

If j is odd, then $2i$ is prime relative to j. If j is even, then i is prime relative to $\frac{j}{2}$.

In the former case, as p takes on every value from 0 to $j-1$, q_p takes on every value from 0 to $j-1$. In the latter case, q_p takes on every even value from 0 to $j-2$ twice. In either case, each element p of the set of integers from 0 to $j-1$ can be paired with another element, p^* (except that one p will remain unpaired if j is odd) such that

$$\left| q_p - q_{p^*} - \frac{j}{2} \right| \leqq 1 \quad \left(\text{or } \left| q_{p^*} - q_p - \frac{j}{2} \right| \leqq 1\right).$$

Then

$$\left| \frac{q_p}{j} - \frac{q_{p^*}}{j} - \frac{1}{2} \right| \leqq \frac{1}{j} \leqq \frac{1}{n^2}$$

and, since

$$\left| \frac{t^2\lambda^2\omega_p}{2\pi} - \frac{q_p}{j} \right| \text{ and } \left| \frac{t^2\lambda^2\omega_{p^*}}{2\pi} - \frac{q_{p^*}}{j} \right|$$

are both $< \frac{7}{n^2}$

$$\left| \frac{t^2\lambda^2\omega_p}{2\pi} - \frac{t^2\lambda^2\omega_{p^*}}{2\pi} - \frac{1}{2} \right| < \frac{15}{n^2}.$$

Now, let $g_p(x) = S[\lambda(t+p)+\mu]$,

let $h_p(x) = S[\lambda(t+p)+\nu]$,

let $\emptyset_p(x) = (x-\alpha-\omega_p)\left(1-\frac{\lambda^2(t+p)^2}{[\lambda(t+p)+\mu]^2}\right)$,

and let $r_p(x) = (x-\alpha-\omega_p)\left(1-\frac{\lambda^2(t+p)^2}{[\lambda(t+p)+\nu]^2}\right)$.

Then $g'_p(x-\alpha-\omega_p-\emptyset_p)=1$ and $h'_p(x-\alpha-\omega_p-r_p)=1$. Using a line of reasoning exactly parallel to the case of β_p, we get

$$\left|\emptyset_p-\frac{2\mu x}{\lambda t}\right| < \frac{7\pi}{2\lambda n t^2}\sqrt{\frac{2x}{\pi}} \text{ and } \left|r_p-\frac{2\nu x}{\lambda t}\right| < \frac{7\pi}{2\lambda n t^2}\sqrt{\frac{2x}{\pi}}.$$

Therefore,

$$\left|\frac{t^2\lambda^2\emptyset_p}{2\pi}-\frac{t^2\lambda^2\emptyset_{p^*}}{2\pi}\right| \text{ and } \left|\frac{t^2\lambda^2 r_p}{2\pi}-\frac{t^2\lambda^2 r_{p^*}}{2\pi}\right|$$

are both $<\frac{7}{2}\sqrt{\frac{2x}{\pi}}$.

Now

$$g_p(x)+\frac{\sin\tau}{[\lambda(t+p)+\mu]^2}=\frac{\sin[\lambda(t+p)+\mu]^2(\alpha+\omega_p+\emptyset_p)}{[\lambda(t+p)+\mu]^2}$$

and $\alpha+\omega_p+\emptyset_p<\frac{3x}{t}$, so

$$\left|g_p(x)+\frac{\sin\tau}{[\lambda(t+p)+\mu]^2}-\frac{\sin\lambda^2t^2(\alpha+\omega_p+\emptyset_p)}{\lambda^2t^2}\right|$$
$$<\frac{3(p+1)x}{t^2}<\frac{3\pi}{2\lambda n t^2}\sqrt{\frac{2x}{\pi}}.$$

Now

$$\left|\frac{t^2\lambda^2(\alpha+\omega_p+\emptyset_p)}{2\pi}-\frac{t^2\lambda^2(\alpha+\omega_{p^*}+\emptyset_{p^*})}{2\pi}-\frac{1}{2}\right|<\frac{15}{n^2}+\frac{7}{2}\sqrt{\frac{2x}{\pi}}<\frac{16}{n^2}.$$

Therefore,

$$\left|\frac{\sin\lambda^2t^2(\alpha+\omega_p+\emptyset_p)}{\lambda^2t^2}-\frac{\sin\lambda^2t^2(\alpha+\omega_{p^*}+\emptyset_{p^*})}{\lambda^2t^2}\right|<\frac{32\pi}{n^2\lambda^2t^2},$$

and

$$\left|g_p(x)+g_{p^*}(x)+\frac{\sin\tau}{[\lambda(t+p)+\mu]^2}+\frac{\sin\tau}{[\lambda(t+p^*)+\mu]^2}\right|$$
$$<\frac{3\pi}{\lambda n t^2}\sqrt{\frac{2x}{\pi}}+\frac{32\pi}{n^2\lambda^2t^2}.$$

Finally,

$$|g_p(x)+g_{p^*}(x)+\frac{2\sin\tau}{\lambda^2t^2}|<\frac{2}{\lambda t^2}\sqrt{\frac{2x}{\pi}}+\frac{3\pi}{\lambda nt^2}\sqrt{\frac{2x}{\pi}}+\frac{32\pi}{n^2\lambda^2t^2}<\frac{33\pi}{n^2\lambda^2t^2}.$$

Similarly,

$$|h_p(x)+h_{p^*}(x)+\frac{2\sin\tau}{\lambda^2t^2}|<\frac{33\pi}{n^2\lambda^2t^2}.$$

Therefore, $|g_p(x)-h_p(x)+g_{p^*}(x)-h_{p^*}(x)|<\frac{66\pi}{n^2\lambda^2t^2}$. Summing over all $p<j$, including the unpaired p:

$$|\sum_{k\in R_u}[S(\lambda k+\mu)-S(\lambda k+\nu)]|<\frac{66\pi j}{n^2\lambda^2t^2}+\frac{1}{\lambda^2t^2}=\frac{66\pi j}{n^2\lambda^2t^2}+\frac{n^2}{n^2\lambda^2t^2}<\frac{67\pi j}{n^2\lambda^2t^2}.$$

Since $j<<t$,

$$<\frac{67\pi j}{n^2\lambda^2t^2}<\frac{68\pi j}{n^2\lambda^2(t+j)^2}<\frac{68\pi}{n^2\lambda^2}\sum_{k\in R_u}\frac{1}{k^2}.$$

Summing over all R_u and applying Lemma 4:

$$|\sum_{\substack{k\notin Q\\ k\geqq b}}[S(\lambda k+\mu)-S(\lambda k+\nu)]|<\frac{68\pi}{n^2\lambda^2}\sum_{\substack{k\notin Q\\ k\geqq b}}\frac{1}{k^2}<\frac{68\pi}{n^2\lambda^2}\sum_{k=b}^{\infty}\frac{1}{k^2}$$

$$<\frac{68\pi}{n^2\lambda^2(b-1)}<\frac{69\pi}{n^2\lambda^2b}<\frac{69x}{n}.$$

Finally,

$$|\sum_{k=b}^{\infty}[S(\lambda k+\mu)-S(\lambda k+\nu)]|<\frac{20x}{\pi n}+\frac{69x}{n}<\frac{76x}{n}.$$

5. Proof of Lemma 2. For all integers $k\geqq 0$, the function

$$f(x)=\frac{[\cos\tau\sin(\lambda k+\mu)^2x]+\sin\tau}{(\lambda k+\mu)^2}$$

has the same period and is tangent at zero to the function

$$g(x)=\frac{\sin[(\lambda k+\mu)^2x+\tau]}{(\lambda k+\mu)^2}.$$

Therefore, $0\leqq\tau\leqq\pi\Rightarrow f(x)\geqq g(x)$ for all x, and $\pi\leqq\tau<2\pi\Rightarrow f(x)\leqq g(x)$ for all x. Clearly, then, in order to prove Lemma 2, it will be sufficient to prove that $\sum_{k=0}^{\infty}\frac{\sin(\lambda k+\mu)^2x}{(\lambda k+\mu)^2}$ has a derivative of $+\infty$ at $x=0$.

First we need a lemma, which we will not prove here:

LEMMA 5. *For all i, and all x s.t. $0<x<\frac{\pi}{2i^2}$,*

$$\frac{\sin i^2 x}{i^2 x} > \frac{2}{\pi}.$$

We will also use Lemma 4 from Section 4.

We introduce the following two functions of x, defined for all $x > 0$: Let $m =$ the least positive integer such that $\frac{\pi}{2(\lambda m + \mu)^2} \leqq x$. Let $a =$ the least integer such that $\lambda a + \mu > \sqrt{2}[\lambda(m-1) + \mu]$. Now we will prove that, given any positive integer n, there exists a positive integer M such that

$$0 < x < \frac{\pi}{2(\lambda M + \mu)^2} \Rightarrow \sum_{k=0}^{\infty} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2} > nx.$$

We will examine the series $\sum_{k=0}^{\infty} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2}$ in four parts; the sum from 0 n, from $n+1$ to $m-1$, from m to $a-1$, and from a to ∞. First, for all integers k, $\lim_{x \to 0} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2 x} = 1$; $x \neq 0$. Therefore, given any positive integer n, $\exists \delta > 0$ s.t. for all k s.t. $0 \leqq k \leqq n$,

$$|x| < \delta \Rightarrow \left| \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2 x} - 1 \right| < \frac{1}{n+1}$$

$$\Rightarrow \frac{nx}{n+1} < \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2}; x \neq 0.$$

This implies that $|x| < \delta \Rightarrow \sum_{k=0}^{n} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2} > nx$. For such a δ, let $M =$ the least positive integer such that

$$\frac{\pi}{2(\lambda M + \mu)^2} < \min\left(\delta, \frac{\pi}{2[14(\lambda n + \mu)]^2}\right)$$

Now $\forall x > 0$, $x < \frac{\pi}{2[\lambda(m-1) + \mu]^2}$. Then, from Lemma 5,

$$k \leqq m - 1 \Rightarrow 0 < x < \frac{\pi}{2(\lambda k + \mu)^2}$$

$$\Rightarrow \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2 x} > \frac{2}{\pi} \Rightarrow \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2} > \frac{2x}{\pi} > \frac{1}{(\lambda m + \mu)^2},$$

$$0 < x < \frac{\pi}{2(\lambda M + \mu)^2} \Rightarrow m - 1 \geqq M \Rightarrow m - 1 > 14n \Rightarrow m - 1 > n + 1,$$

$$0 < x < \frac{\pi}{2(\lambda M + \mu)^2} \Rightarrow \sum_{k=n+1}^{m-1} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2} > \frac{(m-1) - n}{(\lambda m + \mu)^2} \geqq \frac{m - 2n}{(\lambda m + \mu)^2}.$$

Next,

$$\forall x > 0,\ k < a \Rightarrow \lambda k + \mu < \sqrt{2}[\lambda(m-1)+\mu]$$

$$\Rightarrow \frac{\pi}{(\lambda k+\mu)^2} > \frac{\pi}{2[\lambda(m-1)+\mu]^2} > x > 0 \Rightarrow \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} > 0.$$

Therefore, $\forall x > 0,\ m < a \Rightarrow \sum_{k=m}^{a-1} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} > 0.$

Finally,

$$\forall x, \forall k,\ \left|\frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2}\right| \leqq \frac{1}{(\lambda k+\mu)^2},$$

and, from Lemma 4,

$$\begin{aligned}\forall x > 0,\ &\left|\sum_{k=a}^{\infty} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2}\right| \\ &\leqq \sum_{k=a}^{\infty}\left|\frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2}\right| \leqq \sum_{k=a}^{\infty}\frac{1}{(\lambda k+\mu)^2} < \sum_{k=a}^{\infty}\frac{1}{(\lambda k)^2} \\ &< \frac{1}{\lambda^2(a-1)} < \frac{1}{\lambda(\lambda a+\mu-2\lambda)} < \frac{1}{\lambda[\sqrt{2}(\lambda m+\mu)-(\sqrt{2}+2)\lambda]}.\end{aligned}$$

Now, since $m-1 > 14n$, $m > 15$ and $(\sqrt{2}+2) < \frac{\lambda m}{4} < \frac{\lambda m+\mu}{4}$. Therefore

$$\frac{1}{\lambda[\sqrt{2}(\lambda m+\mu)-(\sqrt{2}+2)\lambda]} < \frac{1}{\lambda(\sqrt{2}-\frac{1}{4})(\lambda m+\mu)} < \frac{6}{7\lambda(\lambda m+\mu)}.$$

Also,

$$\frac{m-2n}{(\lambda m+\mu)^2} > \frac{(\lambda m+\mu)-2(\lambda n+\mu)}{\lambda(\lambda m+\mu)^2} > \frac{6}{7\lambda(\lambda m+\mu)}.$$

Therefore,

$$0 < x < \frac{\pi}{2(\lambda M+\mu)^2} \Rightarrow \sum_{k=n+1}^{m-1} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} > \left|\sum_{k=a}^{\infty} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2}\right|,$$

which implies that

$$\sum_{k=n+1}^{m-1} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} + \sum_{k=a}^{\infty} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} > 0.$$

Adding the four parts of the series we get

$$0 < x < \frac{\pi}{2(\lambda M+\mu)^2} \Rightarrow \sum_{k=0}^{\infty} \frac{\sin(\lambda k+\mu)^2 x}{(\lambda k+\mu)^2} > nx,$$

which implies that the right derivative at zero is equal to $+\infty$. But since the right and left derivatives must be equal at zero, the full derivative is $+\infty$.

6. Differentiable point of the Riemann function. We can derive the following corollary from Lemma 1:

COROLLARY 1a. *Let μ, ν, and λ be any integers s.t. $0<\mu<\nu\leqq\lambda$ and let τ be any real number. Then*

$$\sum_{k=0}^{\infty}\left[\frac{\sin[(\lambda k+\mu)^2x+\tau]}{(\lambda k+\mu)^2}-\frac{\sin[(\lambda k+\nu)^2x+\tau]}{(\lambda k+\nu)^2}\right]$$

has a derivative of $\left(\frac{\nu-\mu}{\lambda}\right)\cos\tau$ at zero.

Proof. For $\frac{\pi}{\lambda}\leqq|\tau|\leqq\frac{\pi}{2}$ or $\tau=0$, the corollary follows directly from Lemma 1, since the left derivative of

$$\sum_{k=0}^{\infty}\left[\frac{\sin[(\lambda k+\mu)^2x+\tau]}{(\lambda k+\mu)^2}-\frac{\sin[(\lambda k+\nu)^2x+\tau]}{(\lambda k+\nu)^2}\right]$$

is equal to the right derivative of

$$\sum_{k=0}^{\infty}\left[\frac{\sin[(\lambda k+\mu)^2x-\tau]}{(\lambda k+\mu)^2}-\frac{\sin[(\lambda k+\nu)^2x-\tau]}{(\lambda k+\nu)^2}\right]$$

and $\left(\frac{\nu-\mu}{\lambda}\right)\cos\tau=\left(\frac{\nu-\mu}{\lambda}\right)\cos(-\tau)$. To extend this result to any real value of τ, simply multiply λ, μ, and ν by an appropriate constant.

Now we can consider the differentiability of $\sum_{k=1}^{\infty}\frac{\sin k^2x}{k^2}$ at points of the form $\frac{A\pi}{B}$. First, what are the value and derivative of $\frac{\sin k^2x}{k^2}$ at $\frac{A\pi}{B}$ for given k? Since $\frac{\sin k^2x}{k^2}$ has a cycle of length $\frac{2\pi}{k^2}$, the value and derivative of $\frac{\sin k^2x}{k^2}$ are the same at $\frac{A\pi}{B}$ and $\frac{\pi P_k}{k^2B}$, where $Ak^2\equiv P_k \bmod 2B$. Therefore if, for two positive integers μ and ν, where $\mu<\nu\leqq 2B$, either $P_\nu=P_\mu+B$ or $P_\mu=P_\nu+B$, then $\sum_{k=0}^{\infty}\left(\frac{\sin(2Bk+\mu)^2x}{(2Bk+\mu)^2}+\frac{\sin(2Bk+\nu)^2x}{(2Bk+\nu)^2}\right)$ has a derivative of $\left(\frac{\nu-\mu}{2B}\right)\cos\frac{\pi P_\mu}{B}$ at $\frac{A\pi}{B}$.

The following procedure, then, can be used to determine the existence and value of the derivative of $\sum_{k=1}^{\infty}\frac{\sin k^2x}{k^2}$ at certain points of the form $\frac{A\pi}{B}$:

List the values of P_k for all k from 1 to $2B$. If all the P_k can be paired off into (μ, ν) pairs s. t. $P_\mu + B = P_\nu$, then the derivative exists and is equal to the sum of $(\frac{\nu - \mu}{2B})\cos\frac{\pi P_\mu}{B}$ over all (μ, ν) pairs. If all P_k can be paired off except for certain values of k, and the sum of $\frac{\sin k^2 x}{k^2}$ taken over these values of k is known to be non-differentiable, then $\sum_{k=1}^{\infty}\frac{\sin k^2 x}{k^2}$ is not differentiable at $\frac{A\pi}{B}$. Now we are ready to prove the two theorems stated in Section 1.

THEOREM 1. *The derivative of* $\sum_{k=1}^{\infty}\frac{\sin k^2 x}{k^2}$ *at* $\frac{(2A+1)\pi}{2B+1}$ *is equal to* $\frac{-1}{2}$ *for all integers* A, B.

Proof. Let μ be any positive integer $\leqq 2B+1$ and let

$$(2A+1)\mu^2 \equiv P \bmod 2(2B+1).$$

Then,

$$\begin{aligned}
(2A+1)(2B+1+\mu)^2 &= (2A+1)(4B^2+4B+4B\mu+2\mu+\mu^2+1)\\
&= (2A+1)[(4B^2+2B)+(2B+1)+(4B\mu+2\mu)+\mu^2]\\
&= (2A+1)[2B(2B+1)+2\mu(2B+1)+(2B+1)+\mu^2]\\
&\equiv (2A+1)(2B+1+\mu^2) \bmod 2(2B+1)\\
&= (2A+1)(2B+1)+(2A+1)\mu^2\\
&= 2A(2B+1)+(2B+1)+(2A+1)\mu^2\\
&\equiv (2A+1)\mu^2+(2B+1) \bmod 2(2B+1)\\
&\equiv P+(2B+1) \bmod 2(2B+1).
\end{aligned}$$

Therefore each μ can be paired off with a ν equal to $\mu+2B+1$ and $\sum_{k=1}^{\infty}\frac{\sin k^2 x}{k^2}$ is differentiable at $\frac{(2A+1)\pi}{2B+1}$. Since

$$(2A+1)\mu^2 = (2A+1)(-\mu)^2 \equiv (2A+1)[2(2B+1)-\mu]^2 \bmod 2(2B+1),$$

for each (μ, ν) pair, there is another (μ, ν) pair in which the values of P_μ and P_ν are reversed, except for $\mu = 2B+1$, $\nu = 2(2B+1)$. Therefore, each $\frac{\nu-\mu}{2(2B+1)}\cos\frac{\pi P}{(2B+1)}$ is matched by a $\frac{\mu-\nu}{2(2B+1)}\cos\frac{\pi P}{(2B+1)}$ except for $\mu = 2B+1$, $\nu = 2(2B+1)$, so the derivative of $\sum_{k=1}^{\infty}\frac{\sin k^2 x}{k^2}$ is equal to $\frac{-1}{2}$ at $\frac{(2A+1)\pi}{2B+1}$.

THEOREM 2. $\sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}$ *has no derivative at* $\frac{(2A+1)\pi}{2^N}$ *for all integers* A, N *with* $N \geqq 1$.

Proof. The case of $N=1$ has been proved by Hardy, and the case of $N=2$ follows directly from Lemma 2.

Consider $N \geqq 3$. Let μ be any odd integer $\leqq 2^{N+1}$ and let $(2A+1)\mu^2 \equiv P \bmod 2^{N+1}$. Then,

$$\begin{aligned}(2A+1)(2^{N-1}+\mu)^2 &= (2A+1)(2^{2N-2}+2^N\mu+\mu^2)\\ &\equiv (2A+1)(2^N\mu+\mu^2) \bmod 2^{N+1}\\ &= 2^{N+1}A\mu + 2^N\mu + (2A+1)\mu^2 \equiv 2^N + P \bmod 2^{N+1}.\end{aligned}$$

The even values of μ need not be considered if the function is known to be non-differentiable at points of the form $\frac{(2A+1)\pi}{2^{N-2}}$. Therefore, by induction, $\sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}$ is not differentiable at $\frac{(2A+1)\pi}{2^N}$ for all integers A, N with $N \geqq 1$. This concludes the proofs.

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