Products

Note Title 2/25/2009

Theorem 1. Let $f \in R[a,b]$. Then $f^2 \in R[a,b]$

Proof: We know $|f| \in R$. $f^2 = (|f|)^2$ so we may assume $f \geq 0$. Let M be the sup of f on some interval. Then M^2 is the sup of f^2 on that interval. Similarly with the inf, m. Let P be a partition of $[a,b]$ so that

$$S_P(f) - s_P(f) = \sum (M_i - m_i)\Delta x_i < \epsilon .$$

Then
$$\begin{aligned} S_P(f^2) - s_P(f^2) &= \sum (M_i^2 - m_i^2)\Delta x_i \\ &\leq 2M\left(\sum (M_i - m_i)\Delta x_i\right) \\ &< 2M\epsilon, \end{aligned}$$

where $M = \sup\{f(x) : x \in [a,b]\}$.

Theorem 2. Let $f, g \in R[a,b]$. Then $fg \in R[a,b]$

Proof: $(f \pm g)^2 = f^2 \pm 2fg + g^2 \in R[a,b]$.

Hence $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right] \in R[a,b]$.