Heat Kernel

Note Title 3/10/2009

The solution of the heat equation for a ring of radius 1 and initial temperature $f(\theta)$ is given by

$$u(\theta,t) = \sum_{-\infty}^{+\infty} c_n e^{-n^2kt} e^{in\theta},$$

where $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi) e^{-in\varphi}\, d\varphi$.

We can rewrite this in terms of the heat kernel

$$H(\theta,t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{-n^2kt+in\theta}$$ as follows

$$u(\theta,t) = \int_0^{2\pi} f(\varphi) H(\theta-\varphi,t)\, d\varphi,$$

convolution with the heat kernel.

There is no "simple" expression for $H(\theta,t)$.

[This formula is also correct for the solution of $u_t(\theta,t) = k u_{xx}(\theta,t)$ $u(0,t) = u(\pi,t) = 0$, $u(x,0) = f(x)$, $0 < x < \pi$ and extend f to be odd and 2π-periodic.]