Absolute and Uniform Convergence

Consider the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

We know that $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \log(1+x)$ for $x \in [0,1]$. Later we will prove (Abel's theorem) that the convergence is uniform on $[0,1]$ and hence also on $[0,1)$, the half-open, half-closed interval. It's also easy to see that the convergence is absolute on $[0,1)$,

$$|x| + \frac{|x|^2}{2} + \frac{|x|^3}{3} + \cdots = -\log(1-|x|).$$

However $|x| + \frac{|x|^2}{2} + \frac{|x|^3}{3} + \cdots$ does not converge uniformly on $[0,1)$. If it did,

$$\left\| \frac{|x|^m}{m} + \cdots + \frac{|x|^n}{n} \right\|_{\infty,[0,1)} \longrightarrow 0 \text{ as } n, m \to \infty.$$

But $\left\| \frac{|x|^m}{m} + \cdots + \frac{|x|^n}{n} \right\|_{\infty,[0,1)} = \left\| \frac{|x|^m}{m} + \cdots + \frac{|x|^n}{n} \right\|_{\infty,[0,1]}$.

Hence the series would converge uniformly on $[0,1]$. But the series doesn't converge at all for $x=1$.