# Abel's Lemmas

Note Title 2/18/2009

1. Let $a_n \geq a_{n+1} \geq \cdots \geq 0$, let $s_n = b_n$, $s_{n+1} = b_n + b_{n+1}$, $s_m = b_n + \cdots + b_m$. Suppose $h \leq s_j \leq H$, $j = n, \ldots, m$. Then

(*) $h a_n \leq a_n b_n + \cdots + a_m b_m \leq H a_n$

Proof:

$$a_n b_n + \cdots + a_m b_m = a_n s_n + a_{n+1}(s_{n+1} - s_n) + \cdots + a_{m-1}(s_{m-1} - s_{m-2}) + a_m(s_m - s_{m-1})$$

$$= s_n(a_n - a_{n+1}) + s_{n+1}(a_{n+1} - a_{n+2}) + \cdots + s_{m-1}(a_{m-1} - a_m) + s_m a_m$$

$$\leq H(a_n - a_{n+1} + a_{n+1} - a_{n+2} + \cdots + a_{m-1} - a_m + a_m)$$

$$= H a_n$$

The other inequality is similar.

2. Let $f(x) \geq 0$, $f'(x) \leq 0$, and $f'$ continuous on $[c, d]$. Let $\phi(x)$ be continuous on $[c, d]$. Let

$$\Phi(\xi) = \int_c^\xi \phi(t)\,dt.$$ Suppose $h \leq \Phi(\xi) \leq H$ for $\xi \in [c, d]$. Then

(**) $$h f(c) \leq \int_c^d f(x)\phi(x)\,dx \leq H f(c).$$

Proof: Integrate by parts:

$$\int_c^d f(x)\,\phi(x)\,dx = \int_c^d f(x)\,d\Psi(x) = f(x)\Psi(x)\Big|_c^d - \int_c^d \Psi(x) f'(x)\,dx.$$

$\Psi(c) = 0$, and

$$-\int_c^d \Psi(x) f'(x)\,dx \le -H\int_c^d f'(x)\,dx = H\big(f(c) - f(d)\big).$$

Hence

$$\begin{aligned}\int_c^d f(x)\phi(x)\,dx &\le f(d)\Psi(d) + H\big(f(c) - f(d)\big)\\ &= Hf(c) + f(d)\big(\Psi(d) - H\big)\\ &\le Hf(c).\end{aligned}$$

The proof of the other inequality is similar.