The Riemann-Lebesgue lemma is quite general, but since we only know Riemann integration, I’ll state it in that form.

**Theorem 1.** Let $f$ be Riemann integrable on $[a, b]$. Then

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) \cos(\lambda t) dt = 0$$  \hspace{1cm} (1)  

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) \sin(\lambda t) dt = 0$$  \hspace{1cm} (2)  

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) e^{i\lambda t} dt = 0$$  \hspace{1cm} (3)  

**Proof.** I will prove only the first statement. Since $f$ is integrable, given $\epsilon > 0$, there is a partition

$$\{a = x_0, x_1, \ldots, x_n = b\},$$

so that $\frac{\epsilon}{2} > \int_a^b f - \sum_{i=1}^{n} m_i \Delta x_i \geq 0$, where $m_i$ is the minimum of $f$ on $[x_{i-1}, x_i]$.

But the sum can be written as $\sum_{i=1}^{n} m_i \Delta x_i = \int_a^b g$, where $g = \sum m_i \chi_{[x_{i-1}, x_i]}$, and the inequality takes the form

$$\epsilon/2 > \int_a^b (f - g) \geq 0.$$

Now we use the fact that $f - g \geq 0$ to get

$$\left| \int_a^b f(t) \cos(\lambda t) dt \right| \leq \left| \int_a^b (f(t) - g(t)) \cos(\lambda t) dt \right| + \left| \int_a^b g(t) \cos(\lambda t) dt \right|$$  \hspace{1cm} (4)  

$$\leq \int_a^b (f - g) + \left| (1/\lambda) \sum m_i (\sin(\lambda x_i) - \sin(\lambda x_{i-1})) \right|.$$  \hspace{1cm} (5)  

The function $g$ has been fixed. Take $\lambda$ large enough that

$$\left| (1/\lambda) \sum m_i (\sin(\lambda x_i) - \sin(\lambda x_{i-1})) \right| < \epsilon/2,$$

and we are done.

This proof works nearly verbatim for Lebesgue integration and non-compact intervals.