Here is a list of facts (without proof) about Fourier analysis.

1. Suppose \( f \) is integrable (Riemann or Lebesgue) on \([-\pi, \pi]\) and \( \hat{f}(n) = 0 \) for all \( n \). Then \( f = 0 \) almost everywhere (almost everywhere means except on a set of measure 0). If \( f \) is continuous then \( f \) is identically 0. Briefly 
\[
\hat{f}(n) = 0 \text{ for all } n \implies f = 0 \text{ a.e.}
\]

2. Let \( S_N(x) = \sum_{-N}^{N} \hat{f}(n)e^{inx} \), where \( f \in L^2([-\pi, \pi]) \). Then \( \|f - S_N\|_2 \to 0 \) as \( n \to \infty \).

3. If \( f, g \in L^2([-\pi, \pi]) \) then 
\[
<f, g> = 2\pi \sum_{-\infty}^{\infty} \hat{f}(n)\bar{g}(n).
\]
Hence 
\[
\|f\|^2 = 2\pi \sum_{-\infty}^{\infty} |\hat{f}(n)|^2
\]

4. If \( \sum_{-\infty}^{\infty} |c_n|^2 < \infty \) then there is \( f \in L^2([-\pi, \pi]) \) so that \( \hat{f}(n) = c_n \) (and \( \|f - S_N\|_2 \to 0 \) as \( n \to \infty \)).
(Riesz-Fischer theorem)

5. Let \( R^1 \) be the set of Riemann integrable functions on \([-\pi, \pi]\) and \( R^2 = \{ f : |f|^2 \in R^1 \} \). We have proved \( R^1 \subset R^2 \). Let \( L^1, L^2 \) be defined similarly for Lebesgue integration. It’s a theorem that \( L^2 \subset L^1 \). If \( f \in R^1 \) or \( f \in L^1 \) then \( \hat{f}(n) \) is defined and the following is true (Riemann-Lebesgue lemma)
\[
f \in R^1 \text{ or } f \in L^1 \implies \hat{f}(n) \to 0 \text{ as } n \to \infty
\]

6. Let \( C^{k+} = \{ f \in C^k : f^{(k)} \text{ is piecewise smooth} \} \). (This means that \( f^{(k+1)} \) exists and is continuous except at finitely many points and at those points \( f^{(k+1)} \) has left and right limits.) Note that \( C^{k+1} \subset C^{k+} \).
\[
f \in C^{k+} \implies \hat{f}^{(k+1)}(n) = (in)^{k+1} \hat{f}(n)
\]
Nothing is said here about convergence. Also
\[
f \in C^{k+1} \implies \hat{f}^{(k+1)}(n) = (in)^{k+1} \hat{f}(n)
\]

7. If \( f \in L^1 \) or \( f \in R^1 \) and \( n^{k+1+\epsilon} \hat{f}(n) \to 0 \), where \( \epsilon > 0 \), then \( f \in C^k \). Briefly,
\[
n^{k+1+\epsilon} \hat{f}(n) \to 0 \implies f \in C^k
\]