Fourier Coefficients of a Riemann-Integrable Function

February 23, 2008

This note contains some details on Fourier coefficients of a 2\pi-periodic Riemann-integrable function. The change of variables formula was only stated, not proved in the text. Here is a very simple version that is adequate for our purposes.

**Theorem 1.** Let \( f \) be Riemann integrable on \( I = [a, b] \). Define \( g \) on \( J = [a+c, b+c] \) by \( g(y) = f(y - c) \) for \( y \in J \). Then \( g \) is Riemann integrable and \( \int_I f = \int_J g \). We sometimes write this as
\[
\int_{a+c}^{b+c} f(y - c) dy = \int_a^b f(x) dx
\]

**Proof.** Let \( P_1 = \{x_1, x_2, \ldots, x_n\} \) be a partition of \( I \). Then \( P_2 = \{x_1 + c, x_2 + c, \ldots, x_n + c\} \) is a partition of \( J \) and \( S_{P_1}(f) = S_{P_2}(g) \). Hence it follows that \( g \in R(J) \).

**Theorem 2.** Let \( f \) be piecewise smooth on \( \mathbb{R} \) and periodic with period \( P \). Then
\[
\int_0^P f = \int_c^{c+P} f
\]

**Proof.** Let \( g(y) = \int_y^{y+P} f \). Then \( g \) is continuous and at points, \( y \), of continuity of \( f \), \( g'(y) = f(y + P) - f(y) = 0 \). Hence \( g \) is constant on each subinterval where \( f \) is continuous. This constant has to be the same on every subinterval, since \( g \) is continuous everywhere. What is the constant? It is \( \int_0^P f \).

Now we apply these results to the integral that appears in the proof of Dirichlet’s theorem.

**Corollary 1.**
\[
\int_{-\pi}^{\pi} f(x) D_N(x - x_0) dx = \int_{-\pi}^{\pi} f(y + x_0) D_N(y) dy = \int_{-\pi}^{\pi} f(y + x_0) D_N(y) dy = \int_{-\pi}^{0} f(y + x_0) D_N(y) dy + \int_{0}^{\pi} f(y + x_0) D_N(y) dy
\]