Fourier Coefficients of a Riemann-Integrable Function

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This note contains some details on Fourier coefficients of a 2π-periodic Riemann-integrable function. The change of variables formula was only stated, not proved in the text. Here is a very simple version that is adequate for our purposes.

Theorem 1. *Let f be Riemann integrable on $I = [a, b]$. Define g on $J = [a + c, b + c]$ by $g(y) = f(y - c)$ for $y \in J$. Then g is Riemann integrable and $\int_I f = \int_J g$. We sometimes write this as*

$$\int_{a+c}^{b+c} f(y - c)dy = \int_a^b f(x)dx$$

Proof. Let $\mathcal{P}_1 = \{x_1, x_2, \ldots, x_n\}$ be a partition of I. Then $\mathcal{P}_2 = \{x_1 + c, x_2 + c, \ldots, x_n + c\}$ is a partition of J and $S_{\mathcal{P}_1}(f) = S_{\mathcal{P}_2}(g),\ s_{\mathcal{P}_1}(f) = s_{\mathcal{P}_2}(g)$. Hence it follows that $g \in \mathcal{R}(J)$. □

Theorem 2. *Let f be piecewise smooth on $\mathbf{R}$ and periodic with period P. Then*

$$\int_0^P f = \int_c^{c+P} f$$

Proof. Let $g(y) = \int_y^{y+P} f$. Then g is continuous and at points, y, of continuity of f, $g'(y) = f(y + P) - f(y) = 0$. Hence g is constant on each subinterval where f is continuous. This constant has to be the same on every subinterval, since g is continuous everywhere. What is the constant? It is $\int_0^P f$. □

Now we apply these results to the integral that appears in the proof of Dirichlet's theorem.

Corollary 1.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)D_N(x - x_0)dx &= \int_{-\pi-x_0}^{\pi-x_0} f(y + x_0)D_N(y)dy = \int_{-\pi}^{\pi} f(y + x_0)D_N(y)dy \\ &= \int_{-\pi}^{0} f(y + x_0)D_N(y)dy + \int_0^{\pi} f(y + x_0)D_N(y)dy\end{aligned}$$