Constant Coefficient ODEs

February 15, 2006

This is a careful discussion of linear constant coefficient homogeneous ODEs. A reference for this is [1], *Analysis, An Introduction* by Richard Beals.

This note will derive the following result.

Theorem 1. *Let*

$$a_n f^{(n)}(x) + \dots a_1 f'(x) + a_0 f(x) = 0 \tag{1}$$

be a linear constant coefficient homogeneous ODE. Let $r_1, \dots, r_m$ *be the (complex) roots of the characteristic equation* $p(r) = a_n r^n + \cdots + a_1 r + a_0 = 0$ *with multiplicities* $k_1, \dots, k_m$. *Then the solution is of the form*

$$\sum_{j=1}^{m} p_j(x) \exp(r_j x) \tag{2}$$

where p_j *is a polynomial of degree* $k_j - 1$.

First we prove the following result (which defines $\exp(x)$).

Theorem 2. *The solution of the initial value problem*

$$f'(x) = af(x), f(0) = \alpha \tag{3}$$

exists and is unique. It is given by the power series

$$\alpha \sum_{0}^{\infty} \frac{(ax)^n}{n!} = \alpha \exp(ax) \tag{4}$$

which has an infinite radius of convergence. In this equation, α *and* a *may be complex numbers.* $\exp(x)$ *satisfies the following identity*

$$\exp(x + y) = \exp(x) \exp(y) \tag{5}$$

Proof. It is easy to see that f is infinitely differentiable. The initial value data determine that $f^{(k)}(0) = \alpha a^k$. It is also easy to see that the Taylor series of the solution of (3) is given by (4); and it is also easy to see (by using the ratio test) that the radius of convergence is infinite and $\alpha \exp(ax)$ satisfies (3). This is the existence. Equation (5) is proved by using the binomial theorem and the Cauchy product formula, which is valid since the series converges absolutely. This proves that $\exp(x)$ is never 0 and that $\exp(-x) = 1/\exp(x)$.

Now for uniqueness. Let f be any solution of (3). Let $g(x) = \exp(-ax)f(x)$. Then $g'(x) = \exp(-ax)(-af(x) + af(x)) = 0$ and $g(0) = \alpha$, so $f(x) = \alpha \exp(ax)$. □

Next we have the following extension of Theorem 2. Let $D = \frac{d}{dx}$.

Theorem 3. *Let f satisfy*

$$(D - aI)^k f(x) = 0. \tag{6}$$

Then

$$f(x) = p(x) \exp(ax) \tag{7}$$

where p is a polynomial of degree $k - 1$.

Proof. Let $g(x) = \exp(-ax)f(x)$. Then $Dg(x) = \exp(-ax)(D - aI)f(x)$ and $D^2 g(x) = \exp(-ax)(D - aI)^2 f(x)$, etc. Hence $D^k g(x) = 0$, so $g(x) = p(x)$, where p is a polynomial of degree $k - 1$. □

Proof. (of Theorem 1) Let

$$p(x) = \prod_{j=1}^{m} (x - r_j)^{k_j}$$

$$a_j(x) = \frac{p(x)}{(x - r_j)^{k_j}}.$$

The set of a_j's has no common factor so there are polynomials b_j so that

$$1 = b_1 a_1 + b_2 a_2 + \cdots + b_m a_m.$$

Hence

$$\begin{aligned} u =& b_1(D)a_1(D)u + b_2(D)a_2(D)u + \cdots + b_m(D)a_m(D)u \\ =& u_1 + u_2 + \cdots + u_m. \end{aligned}$$

Now $(D - r_j I)^{k_j} u_j = b_j(D)p(D)u = 0$ so $u_j(x) = p_j(x) \exp(r_j x)$ with the degree of p_j equal to $k_j - 1$. □

References

1. Richard Beals, Analysis, An Introduction, Cambridge University Press (2004), p. 220.