TOPOLOGY

1. Introduction

By now, we've seen many uses of property of continuity. It is a fairly general property, encompassing the majority of functions used in calculus, yet it is sufficient condition for the use of the intermediate and extreme value theorems. Moreover, it is a necessary condition for differentiability. In light of the central role continuity plays in calculus, we seeks to study continuous functions further.

There are various alternative characterizations of continuity.

Proposition 1.1. *Let $f : A \to B$, $A \in \mathbb{R}^n$, $B \in \mathbb{R}^n$. Then the following are equivalent:*

- *f is continuous*
- *f takes convergent sequences to convergent sequences*
- *For all open $U \in B$, $f^{-1}(U)$ is open in A*
- *For all closed $V \in B$, $f^{-1}(V)$ is closed in A*

Exercise 1.2. Prove Proposition 1.1.

The third statement on the list is the motivation for the definition of a topology, and the more general definition of continuity.

Definition 1.3. Let X be a set. Let T be a collection of subsets of X such that

- $\emptyset \in T$, and $X \in T$
- If $U_1, ..., U_n$ are elements of T, then $\cap_{i=1}^n U_i$ is an element of T.
- If $U_\alpha \in T$ for all $\alpha \in A$, then $\cup_{a \in A}$ is an element of T.

Then T is called a *topology* on X, and the pair (X, T) is said to be a *topological space*. The elements U of T are reffered to as the *open sets* in X. The complement of an open set is called a *closed set*.

In light of Proposition 1.1, we make the follwing definitions.

Definition 1.4. Suppose that (X, T) and (Y, S) are two topological spaces. A map $f : X \to Y$ is said to be continuous if for all open sets $V \in Y$, $f^{-1}(V)$ is an open set in X. If in addition f is bijective (that is, one to one and onto) and f^{-1} is continuous, f is called a *homeomorphism* and (X, T) and (Y, S) are said to be *homeomorphic* as topological spaces.

One way to think of homeomorphisms is as functions that "mold" spaces, without puncturing or closing up holes. Intuitively, this would lead us to expect that, say, an annulus is not homeomorphic to a disk (this is indeed true, but takes a fair amount of work to prove).

2. Examples

$\mathbb{R}^n$ is a topological space, where a set U is said to be open in the topological sense if it is open in the usual sense. This is called the *metric topology* on $\mathbb{R}^n$. This terminology comes from the fact that this space has the topology induced by the

Euclidean metric–that is, a set U is open if for each point $p \in U$, there is some $\epsilon > 0$ such that the ball of radius ϵ centered at p is entirely contained in U (our definition of a ball uses the Euclidean metric $d(x, y) = \sqrt{x^2 + y^2}$, but we could have defined a ball in terms of another metric). Similarly, one can define the metric topology on $\mathbb{Q}$ or $\mathbb{C}$.

Given any set X, we can define the *discrete topology* by declaring *every* subset of X to be open.

On any given set X, we can also define $\emptyset$ and X to be the only open subsets of X. This is known as the *trivial topology* on X.

Another topology that can be defined on any set X (finite or infinite) is the *cofinite topology*. In the cofininte topology, open sets are defined to be those subsets $U \subset X$ such that the complement of U in X is finite (alternatively, the closed sets are the finite sets). Note that if X is finite, the discrete and cofinite topologies on X are equivalent.

Exercise 2.1. Show that the topologies discussed above satisfy the definition.

3. Definitions

Many of the set properties discussed in Chapter 1 of Folland's text can also be defined in the more general setting of topological spaces.

Definition 3.1. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in a topological space (X, T). Then the sequence is said to converge to a point $p \in X$ if for every open set $U \subset X$ with $p \in U$, there exists some $K_U \in \mathbb{N}$ such that $k > K_U$ implies $x_k \in U$.

Exercise 3.2. Show that in $\mathbb{R}^n$ with the metric topology, the above definition is equivalent to the one given in the text.

Definition 3.3. A topological space (X, T) is said to be *disconnected* if it is X is the union of two disjoint open sets. X is connected if it is not disconnected.

Definition 3.4. Let (X, T) be a topological space. Then the *interior* of a set $A \subset X$ is the union of all open sets contained in A, and the *closure* of A is the intersection of all closed sets containing A.

For a discussion of compactness, see page 32 in your text.

Remarkably, the Extreme and Intermediate Value Theorems still hold when the domain is a topological space, and when the definition of compactness and connectedness are replaced with those given above.

4. The Subspace Topology

Given a topological space (X, T) and a subset $A \subset X$, we would like to place a topology on A that is "nice" with respect to the topology on X. One property of continuity that holds for maps between real spaces is this: If $A \subset \mathbb{R}^n$, then a function $f : \mathbb{R}^m \to A$ is continuous if and only if the composite map $i_A \circ f$ is continuous (here i_A denotes the inclusion map from A into X). If we consider arbitrary topological spaces, it turns out that this property uniquely determines a topology on A.

Definition 4.1. Let (X, T) be a topological space, and let A be a subset of X. Define a topology T_A on A by declaring a set $V \subset A$ to be open if and only if

$V = U \cap A$ for some open set $U \subset X$. Then T_A is called the *subspace topology* on A.

Exercise 4.2. Show that the subspace topology is indeed a topology.

Using the subspace topology, we can make precise the meaning of "boundary", as used in Stokes' Theorem. If we consider the surface S given in Stokes' Theorem to have the subspace topology inherited from $\mathbb{R}^3$, the "boundary" of S is the set of all $x \in S$ such that there exists a homeomorphism f from an open set $U \subset S$ containing x to an open subset of $\mathbb{H}^2$ such that the image of x under f lies on the x axis ($\mathbb{H}^2$ denotes the upper half plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$). Intuitively, this means that the boundary of S consists of the points that are on the "edge" of an open set in S.

5. Applications

One of the main questions in topology is this: what spaces are homeomorphic to one another?

Exercise 5.1. Show that any two compact intervals (in the subspace topology) are homeomorphic, and that any two open intervals are homeomorphic.

However, the unit open interval U is *not* homeomorphic to the unit closed interval V. Recall that continuous functions take compact sets to compact sets, so any continuous map $f : V \to U$ cannot be surjective.

Exercise 5.2. Is the half open interval $W = (0, 1]$ homeomorphic to U? To V?

Exercise 5.3. Is the closed unit interval in $\mathbb{R}$ homeomorphic to the closed unit square in $\mathbb{R}^2$?

6. References

Bert Mendelson's "Introduction to Topology" gives a light introduction to the subject. The book is fairly short, but explains the basic concepts. James Munkres' "Topology" is the standard undergraduate text; it is used in Math 441. John Lee's "Introduction to Topological Manifolds" gives a discussion more focused on the theory of manifolds, and is the textbook for Math 544.