If we make some assumptions on the coefficients we can say some very interesting things about a trigonometric series. These statements are best made in the context of the Lebesgue integral. Here are a few of the differences between the Lebesgue \((L)\) and Riemann integral \((R)\). The usual notation in the Lebesgue case is the following. By \(f \in L^p(I)\) we mean that \(f^p\) is Lebesgue integrable on \(I\), which might be an infinite interval. By definition this is equivalent to \(|f|^p \in L^1(L)\). This is similar to saying that the only kind of convergence we will discuss is absolute convergence. Here are a few facts. Let \(I = [a, b]\) be a compact interval.

1. Let \(p \geq 1\). \(f \in L^p(I) \implies f \in L^1(I), \ (L^p \subset L^1). \) But \(L^1 \not\subset L^p, \) if \(p > 1.\)

2. \(f \in L^2(I) \iff \sum |\hat{f}(n)|^2 < \infty. \) This is the Riesz-Fischer Theorem.

Riemann integration is quite different. For example, we know that \(f \in R(I) \implies f^2 \in R(I). \) This is the opposite of the Lebesgue case. Also, such functions as \(x^{-1/2}\) are not Riemann-integrable since they are not bounded, but they are Lebesgue integrable. In the following theorem, a Fourier series is a Lebesgue-Fourier series, not a Riemann-Fourier series.

Here is a list of results (taken from [1] and [2]). Let \(b_1 \geq b_2 \geq \cdots \geq 0, \lim_{n \to \infty} b_n = 0.\)

**Theorem 1.** Let

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{1}
\]

1. The following are equivalent

   (a) \(nb_n < K\) is independent of \(n;\)

   (b) \(\sum_{n=1}^{N} b_n \sin nx| < M, \) independent of \(N;\)

   (c) \((1)\) is the Fourier series of a bounded function;

   (d) \(f\) is bounded.

2. The following are equivalent

   (a) \(\lim_{n \to \infty} nb_n = 0;\)

   (b) \((1)\) converges uniformly;

   (c) \((1)\) is the Fourier series of a continuous function;

   (d) \(f\) is continuous.

3. The following are equivalent
(a) $\sum_{1}^{\infty} \frac{b_n}{n} < \infty$;

(b) (1) converges in $L^1([-\pi, \pi])$;

(c) (1) is the Fourier series of a function in $L^1([-\pi, \pi])$;

(d) $f \in L^1([-\pi, \pi])$.

References

1. Frank Jones, Lebesgue Integration on Euclidean Space, Jones and Bartlett, 1993.