Special Sine Series

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If we make some assumptions on the coefficients we can say some very interesting things about a trigonometric series. These statements are best made in the context of the Lebesgue integral. Here are a few of the differences between the Lebesgue (L) and Riemann integral (R). The usual notation in the Lebesgue case is the following. By $f \in L^p(I)$ we mean that f^p is Lebesgue integrable on I, which might be an infinite interval. By definition this is equivalent to $|f|^p \in L^1(L)$. This is similar to saying that the only kind of convergence we will discuss is absolute convergence. Here are a few facts. Let $I = [a, b]$ be a compact interval.

1. Let $p \geq 1$. $f \in L^p(I) \Longrightarrow f \in L^1(I)$, ($L^p \subset L^1$). But $L^1 \not\subset L^p$, if $p > 1$.
2. $f \in L^2(I) \iff \sum |\widehat{f}(n)|^2 < \infty$. This is the Riesz-Fischer Theorem.

Riemann integration is quite different. For example, we know that $f \in R(I) \Longrightarrow f^2 \in R(I)$. This is the opposite of the Lebesgue case. Also, such functions as $x^{-1/2}$ are not Riemann-integrable since they are not bounded, but they are Lebesgue integrable. In the following theorem, a Fourier series is a Lebesgue-Fourier series, not a Riemann-Fourier series.

Here is a list of results (taken from [1] and [2]). Let $b_1 \geq b_2 \geq \cdots \geq 0$, $\lim_{n\to\infty} b_n = 0$.

Theorem 1. *Let*

$$f(x) = \sum_1^\infty b_n \sin nx \tag{1}$$

1. *The following are equivalent*
   (a) $nb_n < K$ *is independent of* n*;*
   (b) $|\sum_1^N b_n \sin nx| < M$*, independent of* N*;*
   (c) *(1) is the Fourier series of a bounded function;*
   (d) f *is bounded.*
2. *The following are equivalent*
   (a) $\lim_{n\to\infty} nb_n = 0$*;*
   (b) *(1) converges uniformly;*
   (c) *(1) is the Fourier series of a continuous function;*
   (d) f *is continuous.*
3. *The following are equivalent*

(a) $\sum_1^\infty \frac{b_n}{n} < \infty$*;*

(b) (1) converges in $L^1([-\pi, \pi])$*;*

(c) (1) is the Fourier series of a function in $L^1([-\pi, \pi])$*;*

(d) $f \in L^1([-\pi, \pi])$*.*

References

1. Frank Jones, Lebesgue Integration on Euclidean Space, Jones and Bartlett, 1993.
2. A. Zygmund, Trigonometric Series, Cambridge Mathematical Library, Cambridge University Press, 2002.