This note will derive the following result.

**Theorem 1.** Let \( f(x_1, \ldots, x_n) \) be a differentiable function. Let \( S = \{ x_{n+1} = f(x_1, \ldots, x_n) \} \subset \mathbb{R}^{n+1} \). Then the \( n \)-dimensional area of \( S \) is

\[
A(S) = \int (1 + |\nabla f|^2)^{1/2}.
\]

This theorem will rely on the following formula.

**Theorem 2.** Let \( \Pi \) be the \( m \)-dimensional parallelootope in \( \mathbb{R}^n \) with coterminous edges given by the vectors \( v_1, \ldots, v_m \). Then the \( m \)-dimensional area of \( \Pi \) is

\[
A(\Pi) = (\det(C))^{1/2},
\]

where \( C \) is the \( m \times m \) matrix with entries \( v_i \cdot v_j \).

**Proof.** (of Theorem 2) Notice that we can write \( C \) in the following form.

\[
C = V^T V, \quad V = [v_1, \ldots, v_m]
\]

where \( V \) is the \( n \times m \) matrix matrix whose columns are the column vectors \( v_1, \ldots, v_m \). If \( \Pi \) is rectangular (the \( v_i \) are orthogonal), the matrix \( C \) is diagonal and the diagonal entries are \( |v_i|^2 \); so the result is true in this case. \( C \) is symmetric and it is easy to see it is positive semi-definite, since

\[
\sum v_i \cdot v_j x_i x_j = \left| \left\langle \sum_{i=1}^{m} x_i v_i \right\rangle \right|^2.
\]

Hence \( \det C \geq 0 \). (If the vectors are linearly dependent (\( \Pi \) is degenerate), then \( \sum_{i=1}^{m} x_i v_i = 0 \) for some choice of the \( x_i \) and \( \det C = 0 \).) So assume the \( v_i \) are linearly independent. Let’s modify \( v_1 \) by subtracting multiples of \( v_2, \ldots, v_m \) to make it orthogonal to each of \( v_2, \ldots, v_m \) This requires solving the equations

\[
\sum_{j=2}^{m} a_j v_j \cdot v_k = v_1 \cdot v_k, \quad k = 2, \ldots, m.
\]

Since \( v_2, \ldots, v_m \) are linearly independent, this system has a unique solution. If we let \( \overline{v}_1 = \sum_{j=2}^{m} a_j v_j \) we get a new parallelootope with the same area. The new matrix \( \overline{C} \) is related to the original matrix \( C \) by the formula

\[
\overline{C} = A^T C A,
\]
where $A$ is the matrix that differs from the identity matrix by having the last $n - 1$ entries in the first column replaced by $a_2, a_3, \ldots, a_n$. The determinant of $A$ is 1, so $\det(C) = \det(C)$. Eventually this process produces a rectangular parallelotope $\Pi$ with the same area as $\Pi$ and thus

$$A(\Pi) = A(\Pi) = (\det(C))^{1/2} = (\det(C))^{1/2}$$

(5)

Theorem 1 is a consequence of the following lemma. To simplify notation we use vertical bars to denote determinant, $|A| = \det(A)$.

**Lemma 1.** In the statement of the lemma, $d$ denotes the determinant.

$$1 + a_1^2 + a_2^2 + \cdots + a_n^2 = \begin{vmatrix} 1 + a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & 1 + a_2^2 & a_2a_3 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1 + a_n^2 \end{vmatrix} = d$$

(6)

**Proof.** The proof is by induction on $n$.

$$d = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2a_1 & 1 + a_2^2 & a_2a_3 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1 + a_n^2 \end{vmatrix} + a_1 \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2a_1 & 1 + a_2^2 & a_2a_3 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1 + a_n^2 \end{vmatrix}$$

(7)

Now by subtracting appropriate multiples of the first row from the other rows in the second determinant we get

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2a_1 & 1 + a_2^2 & a_2a_3 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & 1 + a_n^2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = a_1$$

(8)

Hence $d = (1 + a_1^2 + \cdots + a_n^2) + a_1^2$.

**Proof.** (of Theorem 1). According to the definition of area, the area of the surface is

$$A(S) = \int A(\Pi)dx_1dx_2\ldots dx_n$$

(9)

Where $\Pi$ is the parallelootope spanned by the column vectors $[1, 0, 0, \ldots, f_{x_1}]^T, [0, 1, 0, \ldots, f_{x_2}]^T, \ldots, [0, 0, \ldots, 1, f_{x_n}]^T$. Theorem 2 and Lemma 1 tell how to compute $A(\Pi)$ and the result is $A(\Pi) = (1 + f_{x_1}^2 + f_{x_2}^2 + \cdots f_{x_n}^2)^{1/2} = (1 + |\nabla f|^2)^{1/2}$.

\[\square\]