

# Areas of Hypersurfaces

December 21, 2005

This note will derive the following result.

**Theorem 1.** *Let  $f(x_1, \dots, x_n)$  be a differentiable function. Let  $S = \{x_{n+1} = f(x_1, \dots, x_n)\} \subset \mathbf{R}^{n+1}$ . Then the  $n$ -dimensional area of  $S$  is*

$$A(S) = \int (1 + |\nabla f|^2)^{1/2}. \quad (1)$$

This theorem will rely on the following formula.

**Theorem 2.** *Let  $\Pi$  be the  $m$ -dimensional parallelotope in  $\mathbf{R}^n$  with coterminous edges given by the vectors  $v_1, \dots, v_m$ . Then the  $m$ -dimensional area of  $\Pi$  is*

$$A(\Pi) = (\det(C))^{1/2}, \quad (2)$$

where  $C$  is the  $m \times m$  matrix with entries  $v_i \cdot v_j$ .

*Proof.* (of Theorem 2) Notice that we can write  $C$  in the following form.

$$C = V^T V, \quad V = [v_1, \dots, v_m] \quad (3)$$

where  $V$  is the  $n \times m$  matrix whose columns are the column vectors  $v_1, \dots, v_m$ . If  $\Pi$  is rectangular (the  $v_i$  are orthogonal), the matrix  $C$  is diagonal and the diagonal entries are  $|v_i|^2$ ; so the result is true in this case.  $C$  is symmetric and it is easy to see it is positive semi-definite, since

$$\sum v_i \cdot v_j x_i x_j = \left| \left( \sum_1^m x_i v_i \right) \right|^2.$$

Hence  $\det C \geq 0$ . (If the vectors are linearly dependent ( $\Pi$  is degenerate), then  $\sum_1^m x_i v_i = 0$  for some choice of the  $x_i$  and  $\det C = 0$ .) So assume the  $v_i$  are linearly independent. Let's modify  $v_1$  by subtracting multiples of  $v_2, \dots, v_m$  to make it orthogonal to each of  $v_2, \dots, v_m$ . This requires solving the equations

$$\sum_2^m a_j v_j \cdot v_k = v_1 \cdot v_k, \quad k = 2, \dots, m.$$

Since  $v_2, \dots, v_m$  are linearly independent, this system has a unique solution. If we let  $\bar{v}_1 = \sum_2^m a_j v_j$  we get a new parallelotope with the same area. The new matrix  $\bar{C}$  is related to the original matrix  $C$  by the formula

$$\bar{C} = A^T C A, \quad (4)$$

where  $A$  is the matrix that differs from the identity matrix by having the last  $n - 1$  entries in the first column replaced by  $a_2, a_3, \dots, a_n$ . The determinant of  $A$  is 1, so  $\det(\bar{C}) = \det(C)$ . Eventually this process produces a rectangular parallelotope  $\bar{\Pi}$  with the same area as  $\Pi$  and thus

$$A(\bar{\Pi}) = A(\Pi) = (\det(\bar{C}))^{1/2} = (\det(C))^{1/2} \quad (5)$$

□

Theorem 1 is a consequence of the following lemma. To simplify notation we use vertical bars to denote determinant,  $|A| = \det(A)$ .

**Lemma 1.** *In the statement of the lemma,  $d$  denotes the determinant.*

$$1 + a_1^2 + a_2^2 + \dots + a_n^2 = \begin{vmatrix} 1 + a_1^2 & a_1 a_2 & a_1 a_3 & \dots & a_1 a_n \\ a_2 a_1 & 1 + a_2^2 & a_2 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & 1 + a_n^2 \end{vmatrix} = d \quad (6)$$

*Proof.* The proof is by induction on  $n$ .

$$d = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ a_2 a_1 & 1 + a_2^2 & a_2 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & 1 + a_n^2 \end{vmatrix} + a_1 \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 a_1 & 1 + a_2^2 & a_2 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & 1 + a_n^2 \end{vmatrix} \quad (7)$$

Now by subtracting appropriate multiples of the first row from the other rows in the second determinant we get

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 a_1 & 1 + a_2^2 & a_2 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & 1 + a_n^2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = a_1 \quad (8)$$

Hence  $d = (1 + a_2^2 + \dots + a_n^2) + a_1^2$ . □

*Proof.* (of Theorem 1). According to the definition of area, the area of the surface is

$$A(S) = \int A(\Pi) dx_1 dx_2 \dots dx_n \quad (9)$$

Where  $\Pi$  is the parallelotope spanned by the column vectors

$[1, 0, 0, \dots, f_{x_1}]^T, [0, 1, 0, \dots, f_{x_2}]^T, \dots, [0, 0, \dots, 1, f_{x_n}]^T$ . Theorem 2 and Lemma 1 tell how to compute  $A(\Pi)$  and the result is  $A(\Pi) = (1 + f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2)^{1/2} = (1 + |\nabla f|^2)^{1/2}$ . □