Surface Area and the Cylinder Area Paradox

Frieda Zames

Frieda Zames is Assistant Professor of Mathematics at New Jersey Institute of Technology where she has been employed since 1966. For the past three years, she has also been working with a federally funded program devising curricula and teaching remedial mathematics to educationally deprived adults in Newark. She earned the Ph.D. in Mathematics Education at New York University in 1972.

Introduction. In 1890, H. A. Schwartz astounded the mathematical world by publishing an example which invalidated the accepted definition of surface area [4]. The "cylinder area paradox" is the name given to this insight that opened the floodgates of mathematical inquiry to investigations into surfaces and surface areas [1].

The primary reason that mathematicians were led astray was the assumption that there is a direct analogy between arc length and surface area. In 1948, Tibor Rado [2] stated: "The present status of the theory supports the view that far-reaching analogies do exist. But the analogies lie deep while the discrepancies are conspicuous . . . ." It seems preferable to indicate the incorrect analogy assumed before illustrating the paradox that pointed up the inconsistencies of an uncritical acceptance.

The Assumption. The length $L(C)$ of a finite curve C will be defined in the usual way. Assume C_i is any sequence of inscribed polygonal arcs such that $C_i \to C$ uniformly. If P is an inscribed polygonal curve, then $L(P) \leqslant L(C)$. Also, if $\epsilon > 0$ is given, there exists a polygonal curve P_ϵ such that $L(P_\epsilon) > L(C) - \epsilon$. These two intuitive ideas show that

$$L(C) = \sup_P L(P)$$

where P is taken over all inscribed polygons. It follows from this that there exists a sequence of polygonal approximations C_i where $C_i \to C$ uniformly such that

$$L(C_i) \to L(C).$$

It was this definition that was confounded when an attempt was made to generalize to areas.

The surface area $A(S)$ of a bounded surface was defined in a manner analogous to the definition of the length of a curve given above. As an example, the following definition of the area of a surface S bounded by a curve is similar to that given by J. A. Serrat in 1880 [5]. Let S_i be any sequence of inscribed polyhedral surfaces which converge uniformly to the given surface S, i.e., $S_i \to S$ uniformly. If P is an inscribed polyhedral surface of surface S, then $A(P) \leqslant A(S)$. In addition, if $\epsilon > 0$

is given, there exists a polyhedral surface P_ϵ such that $A(P_\epsilon) > A(P) - \epsilon$. Therefore,

$$A(S) = \sup_P A(P),$$

where P is taken over all inscribed polyhedral surfaces. Thus, it was incorrectly concluded that there exists a sequence of inscribed polyhedrons S_i where $S_i \to S$ uniformly such that

$$A(S_i) \to A(S).$$

This definition was shown to be untenable by H. A. Schwartz in a letter dated December 25, 1880, to A. Genocchi and independently by G. Peano in a class lecture of May 1882 [1].

The Cylinder Area Paradox. Until Schwartz published this paradox, the definition of surface areas just described was the generally accepted one. The importance of this example is that the paradox arose from using the assumed definition on an extremely simple surface, a cylinder [6].

Let S be the lateral surface of a right circular cylinder of height h and radius r. Divide S into m bands by circles lying in planes parallel to the base such that each band has altitude h/m. Select two adjacent bands, as in the figure below, and divide each of the three circles into n congruent arcs such that the endpoints of these arcs are the vertices of the inscribed triangles. The arcs on the top and the bottom will be vertically aligned but the arcs in the middle circle are all displaced through one-half an arc-length. Assume ΔBAC is any one of the congruent inscribed triangles and let D and E be the midpoints of segment BC and arc BC, respectively. Let O be the center of the circle containing arc BC, so that ΔBOC is parallel to the base of the cylinder and assume $\angle BOC = 2\theta$. Thus, $\theta = \pi/n$ and

$$|BC| = 2r \sin\theta = 2r \sin\frac{\pi}{n}.$$

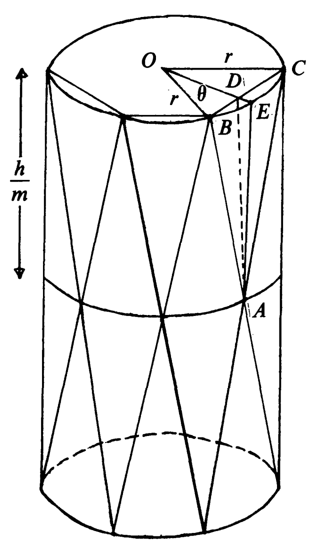

Figure 1.

To find the area of ΔBAC, it is first necessary to apply the Pythagorean theorem to ΔADE in order to calculate the altitude $|AD|$:

$$|DE| = |OE| - |OD| = r - r\cos\theta = r\left(1 - \cos\frac{\pi}{n}\right),$$

and

$$|AD|^2 = |AE|^2 + |DE|^2 = \left(\frac{h}{m}\right)^2 + r^2\left(1 - \cos\frac{\pi}{n}\right)^2,$$

so that

$$\begin{aligned} \text{area}\ (\Delta BAC) &= \frac{1}{2}|BC|\,|AD|, \\ &= \frac{1}{2}\left(2r\sin\frac{\pi}{n}\right)\sqrt{\left(\frac{h}{m}\right)^2 + r^2\left(1 - \cos\frac{\pi}{n}\right)^2}, \\ &= \left(r\sin\frac{\pi}{n}\right)\sqrt{\left(\frac{h}{m}\right)^2 + r^2\left(1 - \cos\frac{\pi}{n}\right)^2}. \end{aligned}$$

In one band of the cylinder between two consecutive parallel circles there are $2n$ of these congruent triangles. Thus, for the m bands, there are $2mn$ such congruent triangles and the sum of their areas is

$$\begin{aligned} A(m, n) &= 2mn\left(r\sin\frac{\pi}{n}\right)\sqrt{\left(\frac{h}{m}\right)^2 + r^2\left(1 - \cos\frac{\pi}{n}\right)^2}, \\ &= 2r\left(n\sin\frac{\pi}{n}\right)\sqrt{h^2 + (mr)^2\left(1 - \cos\frac{\pi}{n}\right)^2}. \end{aligned}$$

It will now be seen that the way in which the limits are taken determine the value of

$$\lim_{m,\, n\to\infty} A(m, n).$$

Case 1. Let $n \to \infty$ first with m held fixed and then let $m \to \infty$.

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} A(m, n)\right]$$

$$= \lim_{m\to\infty}\left[\lim_{n\to\infty} 2r\left(n\sin\frac{\pi}{n}\right)\sqrt{h^2 + (mr)^2\left(1 - \cos\frac{\pi}{n}\right)^2}\right].$$

Each of the following functions can be approximated by the first few terms of its Taylor series expansion, so that

$$\sin\frac{\pi}{n} \approx \frac{\pi}{n} \qquad \text{and} \qquad \cos\frac{\pi}{n} \approx 1 - \frac{\pi^2}{2n^2}.$$

These approximations are sufficiently accurate for evaluating the following limits:

$$\lim_{n\to\infty}\left(n\sin\frac{\pi}{n}\right) = \lim_{n\to\infty} n\left(\frac{\pi}{n}\right) = \pi,$$

and

$$\lim_{n\to\infty}\left(1 - \cos\frac{\pi}{n}\right) = \lim_{n\to\infty} \frac{\pi^2}{2n^2} = 0.$$

These two results imply that

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} A(m, n)\right] = \lim_{m\to\infty} 2r\pi h = 2\pi rh.$$

It is useful to examine this limit geometrically. When $n \to \infty$ first and m remains fixed, the number of triangles in a band increases indefinitely and approaches the surface area of the band. In this case, since the number of bands is considered fixed at this point, the final answer turns out to be independent of m. Thus, the sum of the areas of these triangles approaches the expected value for the lateral surface area.

Case 2. Let $m \to \infty$ first with n held fixed and then let $n \to \infty$.

$$\lim_{n\to\infty}\left[\lim_{m\to\infty} A(m, n)\right]$$

$$= \lim_{n\to\infty}\left[\lim_{m\to\infty} 2r\left(n \sin\frac{\pi}{n}\right)\sqrt{h^2 + (mr)^2\left(1 - \cos\frac{\pi}{n}\right)^2}\;\right]$$

$$= 2r \lim_{n\to\infty}\left[\left(n \sin\frac{\pi}{n}\right)\lim_{m\to\infty}\sqrt{h^2 + (mr)^2\left(1 - \cos\frac{\pi}{n}\right)^2}\;\right]$$

$$= 2r\left[\lim_{n\to\infty}\left(n \sin\frac{\pi}{n}\right)\cdot \infty\right] = \infty.$$

As in Case 1, this limit is interesting to look at geometrically. When $m \to \infty$ first and n is held fixed, the number of bands increase indefinitely while the number of triangles in any band is constant. Also, as $m \to \infty$, for fixed n, $A \to E$ which implies that ΔBAC becomes practically perpendicular to the surface! It is no wonder that the area of a sequence of sums of such triangles approaches infinity.

Case 3. Let m and n approach infinity simultaneously so that $m/n^2 = c$ where $c \geqslant 0$ is a constant. Clearly, $A(m, n) = A(cn^2, n)$.

$$\lim_{n\to\infty} A(cn^2, n) = \lim_{n\to\infty}\left[2r\left(n \sin\frac{\pi}{n}\right)\sqrt{h^2 + (rcn^2)^2\left(1 - \cos\frac{\pi}{n}\right)^2}\;\right]$$

$$= 2r \lim_{n\to\infty}\left(n \sin\frac{\pi}{n}\right)\cdot\left[\lim_{n\to\infty}\sqrt{h^2 + (rc)^2\left(n^2\left(1 - \cos\frac{\pi}{n}\right)\right)^2}\;\right]$$

Using the Taylor series approximation for $1 - \cos(\pi/n)$ as $\pi^2/2n^2$,

$$\lim_{n\to\infty}\left[n^2\left(1 - \cos\frac{\pi}{n}\right)\right] = \frac{\pi^2}{2}.$$

By combining these results, the calculation can now be completed:

$$\lim_{n\to\infty} A(cn^2, n) = 2\pi r\sqrt{h^2 + \frac{r^2\pi^4c^2}{4}}\;.$$

Since $c \geqslant 0$, all answers greater than or equal to $2\pi rh$ are possible and thus

$$\lim_{n\to\infty} A(cn^2, n) \geqslant 2\pi rh.$$

To illustrate this, let $c = 2\sqrt{3}\, h/\pi^2 r$. Consequently,

$$\lim_{n\to\infty} A(cn^2, n) = 2\pi r\sqrt{h^2 + \frac{r^2\pi^4}{4} \cdot \frac{12h^2}{\pi^4 r^2}} = 2\pi r\sqrt{h^2 + 3h^2} = 4\pi rh.$$

Using the original evaluation of

$$A(m, n) = 2r\left(n \sin \frac{\pi}{n}\right)\sqrt{h^2 + (rm)^2\left(1 - \cos \frac{\pi}{n}\right)^2},$$

it is clear that $2\pi rh$ is the minimum value for $\lim_{m,\, n\to\infty} A(m, n)$. The essence of the paradox is the realization that there is no unique answer for $\lim_{m,\, n\to\infty} A(m, n)$. However, $2\pi rh$ is a lower bound for all these limits.

It might be interesting for the reader to prove:

1. $\lim_{n\to\infty} A(cn, n) = 2\pi rh$,
2. $\lim_{n\to\infty} A(cn^3, n) = \infty$,
3. $\lim_{n\to\infty} A(c\sqrt{n}, n) = 2\pi rh$.

Conclusion. Since the publication of Schwartz's paradox became known, a great number of new interpretations for a theory of surface areas have been proposed. Rado [3] points out that "In most cases the idea of approximating the given surface by polyhedrons has been altogether dropped." Lebesgue's definition of surface area devised in 1902 is an exception. According to Lebesgue, the area of a surface is defined as

$$A(S) = \inf[\lim_{i\to\infty} \inf A(S_i)]$$

where the infimum of $A(S_i)$ is taken over all sequences of polyhedral surfaces that converge uniformly to S. This definition of surface area created by Lebesgue circumvented the contradictions posed by the cylinder area paradox and also maintained the surface area concept as basically geometrical. However, the theory of surface area is still surprisingly incomplete although very satisfactory results have been obtained for certain special classes of surfaces [3].

REFERENCES

1. Lamberto Cesari, Surface Area, Princeton University Press, New York, 1956, 24–26.
2. Tibor Rado, Length and Area, Amer. Math. Soc. Colloq. Publication, New York, 1948, 6.
3. ———, On the Problem of Plateau, Chelsea, New York, 1951, 2–18.
4. H. A. Schwartz, Sur une définition erronée de l'aire surface courte, Gesammelte Mathematische Abhandlungen, Berlin, 1890, II, 309–311. This example cited in a letter from H. A. Schwartz was first published in the second mimeographed edition of the Cours by C. Hermite, 1882–1883, 35–36.
5. J. A. Serrat, Cours de Calcul Différentiel et Intégral (2nd ed.), Paris, 1880, 293.
6. G. B. Thomas, Jr., Calculus and Analytic Geometry (4th ed.), Addison-Wesley, Reading Mass., 1968, 568–571.