Products, Sup, Inf, and Absolute Value

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Theorem 1. *Let f, g be integrable. Then $f^+, |f|, fg, \sup(f,g), \inf(f,g)$ are integrable.*

Proof. We first prove that f^+ is integrable. It is easy to check that

$$\mathcal{S}_{\mathcal{P}}(f^+) - s_{\mathcal{P}}(f^+) \leq \mathcal{S}_{\mathcal{P}}(f) - s_{\mathcal{P}}(f).$$

This proves that f^+ is integrable. Since $f = f^+ - f^-$, f^- is integrable. Now it follows that $|f| = f^+ + f^-$ is integrable. Because

$$\inf(f,g) = \frac{1}{2}\left(f + g - |f-g|\right)$$
$$\sup(f,g) = \frac{1}{2}\left(f + g + |f-g|\right)$$

$\sup(f,g)$ and $\inf(f,g)$ are integrable. Next, suppose that $f \geq 0$. Let $m_j = \inf\{f(x) : x \in I_j\}$ where I_j is a subinterval and $M_j = \sup\{f(x) : x \in I_j\}$. Also let $K = \sup\{f(x) : x \in I\}$, where I is the interval of integration. Then $M_j^2 - m_j^2 \leq 2K(M_j - m_j)$. Hence

$$\mathcal{S}_{\mathcal{P}}(f^2) - s_{\mathcal{P}}(f^2) \leq 2K(\mathcal{S}_{\mathcal{P}}(f) - s_{\mathcal{P}}(f)).$$

This proves that if $f \geq 0$ and f is integrable, then f^2 is integrable. Now let f be any integrable function. Then for some c, $g = f - c \geq 0$ and hence $g^2 = f^2 - 2cf + c^2$ is integrable. Hence f^2 is integrable. Finally suppose f and g are integrable. Then $(f+g)^2, (f-g)^2$ are integrable. Hence $fg = \dfrac{1}{4}((f+g)^2 - (f-g)^2)$ is integrable. □

Corollary 1. *Let A, B be measurable sets. Then $A \cup B$ and $A \cap B$ are measurable.*

Proof.

$$\chi_{A\cup B} = \sup(\chi_A, \chi_B), \;\; \chi_{A\cap B} = \inf(\chi_A, \chi_B)$$

□