lim sup

Definition 1. *Suppose the sequence $\{a_n\}$ is bounded.*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k : k \geq n\}).$$

Theorem 1. *There is a unique number a such that:*

Given any $\epsilon > 0$ there are infinitely many n such that $a_n > a - \epsilon$, but only finitely many n such that $a_n > a + \epsilon$ and

$$a = \limsup_{n\to\infty} a_n$$

Proof. Let $\ell = \limsup a_n$. This number is well defined since it is the limit of a decreasing sequence of bounded numbers. We first claim that ℓ satisfies the defining properties of a. Let $M_n = \sup\{a_k : kgen$. Choose N so that $\ell \leq M_n < \ell + \epsilon$. Then there are no numbers a_k with $k > N$ with $a_k > \ell + \epsilon$. Hence there are finitely many n such that $a_n > \ell + \epsilon$. Next If we choose N large enough, $M_n > \ell - \epsilon$ when $n > N$. Hence for each $n > N$ there must be a number $k_n \geq n$ with $a_{k_n} > \ell - \epsilon$. This gives an infinite sequence of numbers a_{k_n} with $a_{k_n} > \ell - \epsilon$. So there is such a number (ℓ).

Suppose there was another such number, call it b. Then for large enough n, $a_n \leq b + \epsilon$ and hence $\ell \leq b + \epsilon$. But also there are infinitely many n so that $a_n > b - \epsilon$. Hence $\ell > b - \epsilon$. But ϵ is arbitrary so $b = \ell$ and the number $a = b = \ell$ in the theorem is unique.

□