Heine-Borel Theorem

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Theorem 1. *$K \subset \mathbb{R}^n$ is compact if and only if every open covering $\{U_\alpha\}$ of K has a finite subcovering $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_s}\}$.*

We first discuss countability.

Definition 1. — *A set X is* countable *if its elements can be put in a seqeuence*

$$X = \{x_1, x_2, \dots, x_k, \dots\}$$

Let $(a_k), (b_k), (c_k)$ be sequences. Then $(a_1, b_1, c_2, a_2, b_2, c_2, \dots)$ is a sequence. Similarly, if $A_k, k = 1, \dots, m$ is a finite set of countable sets $A_1 \cup A_2 \dots \cup A_m$ is a countable set. Moreover, if $(A_1, A_2, \dots)$ is a sequence of countable sets, $A_1 \cup A_2 \cup \dots$ is a countable set. To prove this, let $A_k = \{a_{k1}, a_{k2}, a_{k3}, \dots\}$. Then

$$A_1 \cup A_2 \cup \dots = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \dots\}.$$

Using this we can prove that the set of points $\{(p, q) : p \in \mathbb{Z}, q \in \mathbb{Z}\}$ is a countable set. If we consider only those points with $p > 0, q > 0$ we can prove that the set of positive rational numbers is a countable set, and then we can prove that the set of all rational numbers is a countable set. Finally we can prove the set of all balls in $\mathbb{R}^n$ with rational radii and with centers at points that have rational coordinates is a countable set. Let U be any open set in $\mathbb{R}^n$. Then U is the union of all balls with rational radii and rational coordinates that belong to U.

Now we are ready to prove the Heine-Borel theorem.

Proof. First assume K is compact and $\{U_\alpha\}$ is an open covering. Let $\{B_j\}$ be the set of open balls with rational radii and coordinates such that for each j, $B_j \subset U_{\alpha_j}$ for some α_j. We are including *every* B_j that fits inside some U_α. There are a a countable number of such B_j and we put them in a list $B_1, B_2, \dots$ (maybe a new indexing). They also cover K (each U_α is a union of such balls). Now suppose there is a point of K that is not in B_1. Call it x_1. Then suppose there is a point $x_2 \notin B_1 \cup B_2$, etc. In other words suppose no finite collection of B_j covers K. We have a sequence $x_m \in K$ so that

$$x_m \notin B_1 \cup B_2 \cup \dots \cup B_s, \text{ if } m > s.$$

By compactness of K, there is a subsequence that converges to a point of K, $x_{n_j} \to a \in K$. But $a \in B_t$ for some t. Hence $x_{n_j} \in B_t$ for large enough j. But for large enough j, $n_j > t$ and this is a contradiction.

For the converse, if K is not bounded, $\{B_n = \{x : ||x|| < n\}\}$ is an open covering with no finite sub covering. If K is not closed and $a \notin K, a \in \overline{K}$ then $\{B_n = \{x : ||x - a|| > 1/n\}\}$ is an open subcovering of K with no finite subcovering. □