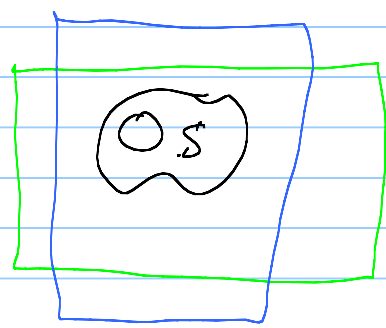
Jordan Content

Note Title 11/21/2009

<u>Theorem</u>: A bounded set S is Jordan measurable if and only if $\mu(\partial S)=0$.

<u>Proof</u>: S is measurable exactly when χ_S is Riemann integrable. I will omit the proof that integrability is independent of the rectangle containing S (this does require proof). So assume



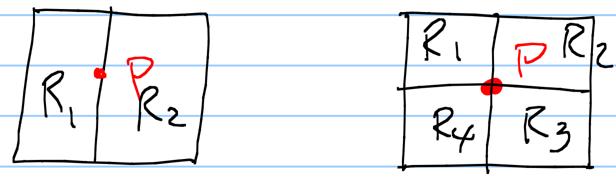
$\bar{S}$ is in the interior of the containing rectangle.

Suppose χ_S is integrable. Then there is a partition so that $S_P(\chi_S) - s_P(\chi_S) < \epsilon$. Let $T = \{ R_{ij} : S \cap R_{ij} \neq \phi, \text{ and } S^c \cap R_{ij} \neq \phi \}$. Then

$$S_P(\chi_S) - s_P(\chi_S) = \sum_{R_{ij} \in T} |R_{ij}| < \epsilon.$$

Let $D = \bigcup_{R_{ij} \in T} R_{ij}$. I claim that $\partial S \subset D$.

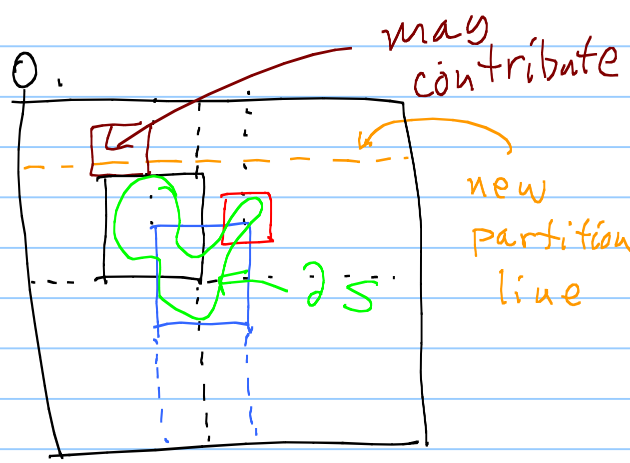
Let $p \in \partial S$. Then if $p \in \text{int}(R_{ij})$, $S \cap R_{ij} \neq \phi$ and $S^c \cap R_{ij} \neq \phi$. So $R_{ij} \in T$. If $p \in \partial R_{ij}$, then p is a corner or edge of R_{ij} and we have one of the following figures



First suppose $p \in S$. Since $p \in \partial S$, every (small) neighborhood of p has a point $q \in S^c$. In the first figure $q \in R_1 \cup R_2$ so either R_1 or $R_2 \in T$. In the second figure $q \in R_1 \cup R_2 \cup R_3 \cup R_4$, so some $R_j \in T$.

Next suppose $p \in S^c$. Then every neighborhood of p contains a point of S, which in either case must be in one of the adjacent rectangles R. So $p \in R$, and $R \cap S \neq \emptyset$. Hence $R \in T$.

In either case $p \in R_{ij}$, where $R_{ij} \in T$. We have proved that $\partial S \subset D = \bigcup_{R_{ij} \in T} R_{ij}$. Since $\sum |R_{ij}| < \epsilon$, we have proved that $\mu(\partial S) = 0$.



Next suppose $\mu(\partial S) = 0$.

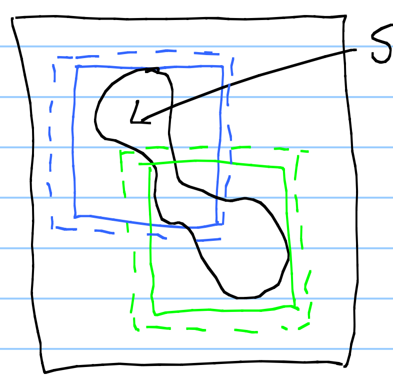
We have a finite union of rectangles R_α with sides parallel to the axes and $\sum |R_\alpha| < \epsilon$. We can create a partition so that these rectangles are unions of rectangles in the partition (see dotted lines for examples. Unfortunately it may NOT happen that

$S_P(\chi_S) - s_P(\chi_S) = \sum |R_\alpha|$. (The brown rectangle) may have points of S and S^c.)

However by adding additional lines we can create a refinement of the partition and with this partition

$S_P(\chi_S) - s_P(\chi_S) < 2\epsilon$, say. So χ_S is integrable.

Or we could replace the original rectangles R_α with slightly larger rectangles R_α', whose interiors cover ∂S (∂S is compact) and then the other rectangles have no points of ∂S. Since rectangles are convex, these other rectangles are entirely contained in S or S^c.



Why doesn't this proof work to show that an integrable function is continuous except on a set of Jordan content 0 (which is not true)?