Note Title 9/30/2009

Cauchy's Inequality

$$\sum a_i^2 \cdot \sum b_i^2 \geq \left(\sum a_i b_i\right)^2$$

In fact:

$$(*)\quad \left\{\begin{aligned} &\sum a_i^2 \cdot \sum b_i^2 - \left(\sum a_i b_i\right)^2 \\ &= \sum_{i<j} (a_i b_j - a_j b_i)^2 = \sum_{i<j} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2 \\ &= \frac{1}{2} \sum_{i,j} (a_i b_j - a_j b_i)^2 \end{aligned}\right.$$

pf: $$\sum_{i,j} (a_i b_j - a_j b_i)^2 = \sum_{i,j} \left[a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i b_i a_j b_j\right]$$

$$= 2 \sum a_i^2 \cdot \sum b_j^2 - 2 \left(\sum a_i b_i\right)^2$$

Q.E.D

Interpretation (in $\mathbb{R}^3$) $A = (a_1, a_2, a_3)$

$A = (b_1, b_2, b_3)$ cross product $A \times B$

$$A \times B = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

$$|A \times B|^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

$$= |A|^2 |B|^2 - (A \cdot B)^2$$

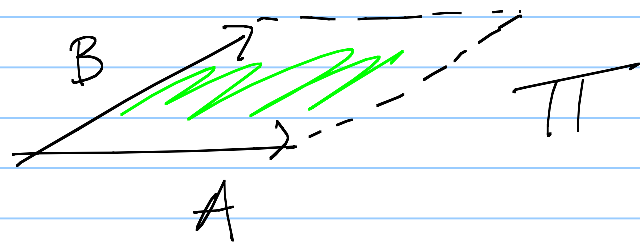
$$= |A|^2 |B|^2 (1 - \cos^2\theta) = |A|^2 |B|^2 \sin^2\theta$$

$|A \times B|$ = area of $\parallel$-ogram spanned by A and B.

This is true in general. The area of the $\parallel$-ogram spanned by A & B is

$$\left(\sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2 \right)^{1/2} = |A||B|\sin\theta = |\Pi|$$

area of Π



Pythagorean theorem: Area of Π = square root of sum of squares of areas of projections of Π onto pairs of axes.