Uniform Continuity

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The concept of uniform continuity was overlooked by mathematicians until Dirichlet and others noticed it is important for the correct statement of some results. I think that in 1862 Dirichlet proved a version of the statement that a continuous function on a compact interval is uniformly continuous. His proof was not published until 1904. Heine learned it from Dirichlet's lectures and apparently published a proof in 1872. I think this was the first use of the concept of compactness, but the term compact was coined much later. So it seems that Heine's proof of the uniform continuity of a continuous function on a compact set is how his name got attached to the Heine-Borel theorem. Lebesgue proved the version of Borel's theorem we are used to. Lebesgue refused to attach Heine's name to it.

Theorem 1. *Let $f : K \to \mathbb{R}^n$ be continuous and suppose K is compact. Then f is uniformly continuous.*

Proof. We define an open covering of K. Fix $\epsilon > 0$. Let $p \in K$. Let $\delta(p) > 0$ be such that if $||q-p|| < 2\delta(p)$ then $||f(q) - f(p)|| < \epsilon/2$. The collection of open balls $B(\delta(p), p)$ covers K. So there is a finite sub cover $\{B_1 = B(\delta_1, p_1), \ldots, B(\delta_k, p_k) = B_k\}$. Suppose $||p - q|| < \delta$, where $\delta = \min\{\delta_1, \ldots, \delta_k\}$. Then $p \in B_j$ for some j since the B_j cover K. By the triangle inequality

$$||q - p_j|| \le ||q - p|| + ||p - p_j|| < \delta + \delta = 2\delta \le 2\delta_j.$$

So

$$||f(p) - f(q)|| \le ||f(p) - f(p_j)|| + ||f(p_j) - f(q)|| < \epsilon/2 + \epsilon/2 = \epsilon.$$

What did we just prove? We proved that if $||p - q|| < \delta$ then $||f(p) - f(q)|| < \epsilon$. No matter where p and q are located. That's uniform continuity. □