Sample Problems

Math 334

The final exam will be held from **8:30 - 10:20 a.m.** on **Monday, December 11**. You may bring one notebook size sheet of paper with notes on *both* sides. There may be homework or example problems on the final exam, in addition to problems similar to the problems on this sheet and the previous sample problem sheets. This is quite a long list of problems. You should do as many as you can. Also you should be prepared to define, state, or use the terms and theorems at the end of this sheet. The final will be comprehensive and will cover through §5.1 in Folland.

1. Let f be defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x\sin(\log x), & \text{if } 0 < x \leq 1. \end{cases}$$

   Prove that f is differentiable on $(0,1]$, but not at 0. Prove that f' is integrable on $[0,1]$ (pick any value for $f'(0)$) and that

$$\int_0^1 f' = 0.$$

2. Let $W \subset \mathbb{R}^n$ be connected and suppose $f : W \to \mathbb{R}$ is continous and only assumes irrational values. Prove that f is constant.
3. Show that the sequence $\sqrt{n^2+n} - n$ converges and find its limit.
4. Give an example of a function $f : [0,1] \to \mathbb{R}$ defined (and finite) at all points of $[0,1]$ such that f is not bounded on any subinterval of $[0,1]$.
5. Suppose $f \in C^2(\mathbb{R})$ and $f''(t) \geq 0$. Prove that the set of critical points of f is a interval.
6. Let $P(x)$ be the parallelogram with vertices

$$(0,0), (f(x), f'(x)), (g(x), g'(x)), (f(x)+g(x), f'(x)+g'(x))$$

   where $f'' = qf, g'' = qg$ and $q(x)$ is some continuous function. Let $A(x)$ be the area of this parallelogram. Show that $A(x)$ is constant.
7. Let I be an interval in $\mathbb{R}$. I might be open, closed, or neither. Let f be a real valued continuous function defined on I. Suppose f has no local maxima or minima in the interior of I. Then prove that f is monotonic.

8. Let $0 < a < b$ and $K > 0$. Prove that

$$\left| \int_a^b \frac{\sin(Kt)}{t} dt \right| \le \pi.$$

9. Let I be an interval in $\mathbb{R}$. I might be open, closed, or neither. Suppose $f : I \to \mathbb{R}$ is strictly increasing. Prove that if the image of f is connected then f is continuous.

10. Let $f(x, y)$ be defined for $0 \le x \le 1, 0 \le y \le 1$ by

$$f(x,y) = \begin{cases} 1 \text{ if } x \text{ is irrational,} \\ 2y \text{ if } x \text{ is rational.} \end{cases}$$

   (a) Prove that $\int_0^1 \left(\int_0^1 f(x,y) dy \right) dx = 1.$

   (b) What can you say about $\int_0^1 \left(\int_0^1 f(x,y) dx \right) dy$?

   (c) Is f integrable?

11. Suppose $a < b < c < d$. Let $I = [a, b],\ J = [c, d],\ R = I \times J$ and let $f(x, y) = |x - y|$, if $x \in I,\ y \in J$. Compute $\int_R f$.

12. Find the volume of the set

$$\{ \left(\frac{x}{1-z} \right)^2 + \left(\frac{y}{1+z} \right)^2 < 1,\ -1 < z < 1 \}$$

13. Let $n > 1$ be a positive integer. Prove that

$$x^n > 1 + n(x - 1),$$

if $x > 1$.

14. (a) Let

$$f(x) = \begin{cases} a, \text{ if } x \le 0, \\ a + x, \text{ if } 0 < x \le 1, \\ a + 1, \text{ if } x > 1. \end{cases}$$

   If f has an antiderivative, find one, and if not prove that none exists.

   (b) Let

$$f(x) = \begin{cases} 1, \text{ if } x \le 0, \\ 2, \text{ if } 0 < x \le 1, \\ 3 \text{ if } x > 1. \end{cases}$$

   If f has an antiderivative, find one, and if not prove that none exists. You may not use Darboux's theorem on this problem.

15. Let $S = \{(x,y,z) : a \le x \le y \le z \le b\}$. Prove that

$$\int_S f(x)f(y)f(z)dxdydz = \frac{1}{6}\left(\int_a^b f\right)^3$$

16. Let f be continuous on $\mathbb{R}$ and suppose $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Suppose $a > 0$.

$$\int_0^1 (f(x+a) - f(x))dx = 0.$$

Use this to prove that there exist x_1, x_2 with $0 \le x_1 < x_2 < 1$ so that

$$f(x_1 + a) = f(x_1)$$
$$f(x_2 + a) = f(x_2).$$

17. (a) Let C_n be the curve $\{(x, \sin\frac{1}{x}) : \frac{1}{n} \le x \le 1\}$. Prove that the length $L(C_n) \to \infty$ as $n \to \infty$.
    (b) Prove that the area of the set $\cup_{n=1}^{\infty} C_n$ is 0.

18. Let u be a function defined on $\mathbf{R}^n$ which is homogeneous of degree k. Prove that $\nabla^2 u$ is homogeneous of degree $k-2$. Let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\mathbf{x}|$. Compute $\nabla^2 r^k$.

19. Compute the area in the first quadrant between the four curves

$$x^3 = a^2 y, x^3 = b^2 y, y^3 = \alpha^2 x, y^3 = \beta^2 x,$$

where $a > b > 0, \alpha > \beta > 0$.

20. Compute the n-dimensional measure of the set:

$$\{(x_1, x_2, \ldots, x_n) : x_j \ge 0, j = 1, \ldots, n,\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n \le n\}$$

21. Important items since the last midterm:

    (a) Inverse function theorem
    (b) Riemann integral and its basic properties
    (c) Jordan measure
    (d) Fubini's theorem
    (e) Change of variables formula
    (f) Arc length
    (g) Line integrals