Connected Sets in $\mathbb{R}$.

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Theorem 1. *The connected subsets of* $\mathbb{R}$ *are exactly intervals or points.*

We first discuss intervals.

Lemma 1. *A set* $X \subset \mathbb{R}$ *is an interval exactly when it satisfies the following property:*
P*: If* $x < z < y$ *and* $x \in X$ *and* $y \in X$ *then* $z \in X$*.*

Proof. If X is an interval **P** is clearly true.

So suppose X is a set that satisfies **P**. Let $a = \inf(X), b = \sup(X)$. We allow $a = -\infty, b = +\infty$. Then X is an interval with endpoints a, b. Let $a < z < b$. Then by definition of $\sup(X)$ there is a point $y \leq b$ of X such that $z < y$. Similarly there is a point $x \geq a$ of X such that $x < z$. Hence $z \in X$ (by **P**). This proves that X is an interval with endpoints a, b. □

Now we prove the theorem.

Proof. First assume I is an interval. If I is disconnected then $I = A \cup B, A \neq \varnothing, B \neq \varnothing, \overline{A} \cap B = \varnothing, A \cap \overline{B} = \varnothing$. Let $a \in A, b \in B$ and assume $a < b$. Let $J = [a, b], A_1 = A \cap J, B_1 = B \cap J$. then $J = A_1 \cup B_1$ is a disconnection of J. Let $c = \sup(A_1)$ Then $c \in \overline{A_1}$ so $c \notin B_1$. But if $c < x \leq b$ then $x \notin A_1$, by definition of $c = \sup(A_1)$. Hence $(x, b] \subset B_1$ and this implies $c \in \overline{B_1}$. So $c \notin A_1$. But c must be in either A_1 or B_1 since $c \in J = A_1 \cup B_1$. $\rightarrow\leftarrow$ (Contradiction).

Now suppose I is not an interval. Then by the lemma there are three points $x < z < y$ with $x \in I, z \notin I, y \in I$. Let $A = \{p \in I, p < z\}, B = \{p \in I, p > z$. Then $I = A \cup B$ is a disconnection of I, so I is not connected. □