Hadamard's Inequality

Let $A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$ be an $n \times n$ matrix, $(A_1, \ldots, A_n)$ are the rows.

Then

(1) $$\left(\det(A)\right)^2 \leq |A_1|^2 \cdots |A_n|^2$$

This can be interpreted as saying that the volume of the parallelotope with edges $A_1, \ldots, A_n$ is no larger than $|A_1| \cdots |A_n|$, the product of the lengths of the rows.

<u>Proof:</u> We will apply the Gram-Schmidt orthogonalization process to A. These row operations preserve determinants. If $A_1 = 0$, the result is true. So suppose $A_1 \neq 0$. Let $B_1 = A_1$

Replace A_2 with

(2) $$A_2 - \frac{\langle B_1, A_2 \rangle}{|B_1|^2} B_1 = B_2.$$

$\langle B_1 \ B_2 \rangle = 0$, so $B_2 \perp B_1$.

If $B_2 \neq 0$, Replace A_3 with

(3) $$A_3 - \frac{\langle B_1, A_3\rangle}{|B_1|^2} B_1 - \frac{\langle B_2, A_3\rangle}{|B_2|^2} B_2 = B_3$$

Then $\langle B_1, B_3\rangle = 0$, and $\langle B_2, B_3\rangle = 0$.

Continue as long as $B_j \neq 0$, $j \leq n-1$. If we ever encounter $B_j = 0$ we are done since in that case the rows of A are linearly dependent. If not, by row operations we have replaced A with a matrix

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$$ with orthogonal rows and the same determinant, $\det A = \det B$. But

$$\det(B B^T) = (\det B)^2 = |B_1|^2 \cdots |B_n|^2$$

and this equals $(\det(A))^2$. Now we relate $|B_j|$ to $|A_j|$. First $|B_1| = |A_1|$.

Next, by (2)

$$A_2 = c_1 B_1 + B_2, \quad \text{where } \langle B_1, B_2\rangle = 0.$$

$$\text{So} \quad |A_2|^2 = |c_1 B_1|^2 + |B_2|^2 \geq |B_2|^2.$$

and we have equality if and only if $c_1 = 0$, and this happens exactly when $\langle A_1, A_2\rangle = 0$. In this case $B_2 = A_2$, nothing changes.

At the next step $A_3 = d_1 B_1 + d_2 B_2 + B_3$, $\langle B_i, B_j \rangle = 0$, and $|A_3|^2 = |d_1 B_1|^2 + |d_2 B_2|^2 + |B_3|^2 \geq |B_3|^2$, with equality exactly when A_1, A_2, A_3 are all orthogonal. Continuing, we get $|B_j|^2 \leq |A_j|^2$ with equality exactly when the rows of A are orthogonal. We have proved (1).

Interpretation: The volume of the parallelotope spanned by $\{A_1, \ldots, A_n\}$ is less than $|A_1| \cdots |A_n|$ with equality exatly when the edges are orthogonal.