Norms on $\mathbb{R}^n$

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Theorem 1. *All norms on $\mathbb{R}^n$ are equivalent (even norms you never heard of). In other words if $|||$ and $||$ are norms then there are positive constants a, b such that*

$$a|||v||| \leq ||v|| \leq b|||v|||, \forall v \in \mathbb{R}.$$

Proof. Let $||$ be any norm. We will prove that there are $a > 0, b > 0$ so that

$$a \|v\| < ||v||_2 < b \|v\| .$$

This will be good enough. First let $v = x_1 e_1 + x_2 e_2 + \ldots x_n e_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. By the triangle inequality

$$\|v\| \leq \sum_j |x_j| \|e_j\| .$$

By Cauchy's inequality, for the inner product $\sum_j |x_j| \|e_j\|$,

$$\sum_j |x_j| \|e_j\| \leq (\sum_j x_j^2)^{1/2} (\sum_j \|e_j\|^2)^{1/2},$$

so

$$a \|v\| \leq ||v||_2, \text{ where } a = 1/((\sum_j \|e_j\|^2)^{1/2}).$$

Next consider the function $\|v\|$ on the set $\|v\|_2 = 1$. The set $K = \{v : ||v||_2 = 1\}$ is compact (closed and bounded. By what we have just proved the function $v \to ||v||$ is continuous on $\mathbb{R}^n$, meaning that if $||v_j||_2 \to 0$ then $\|v_j\| \to 0$. Let $m > 0$ be the minimum of $\|v\|$ on K. Now let $v \neq 0$ be any vector and let $u = v/||v||_2 \in \mathbb{R}^n$. Then $||u||_2 = 1$ so $\|u\| \geq m$. Hence

$$\|v\| \geq m||v||_2, \text{ or } ||v||_2 \leq b \|v\| , \text{ where } b = 1/m.$$

□